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RESEARCH PAPER

EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A FRACTIONAL BOUNDARY VALUE PROBLEM ON A GRAPH

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Abstract

In this paper, the authors consider a nonlinear fractional boundary value problem defined on a star graph. By using a transformation, an equivalent system of fractional boundary value problems with mixed boundary conditions is obtained. Then the existence and uniqueness of solutions are investigated by fixed point theory.

MSC 2010: Primary 34B15; Secondary 34B45

Key Words and Phrases: fractional calculus, boundary value problems, Green's function, differential equations on graphs

1. Introduction

Differential equations on graphs have extensive applications in many areas such as physics, engineering, and ecology. A *graph* G consists of a finite or countably infinite set of *nodes* $V = \{\gamma_i : \ i = 0, 1, \dots\}$ and a set of *edges* E connecting these nodes, i.e., $G = V \cup E$. On each edge, a local coordinate system is assigned with the origin at a node. A differential equation on a graph is a differential equation defined on each edge of G based on the local coordinate system; a problem consisting of such an equation and certain conditions defined at the boundary nodes is called a boundary value problem (BVP) on a graph. For more details about

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differential equations and BVPs on graphs, as well as their applications, the reader is referred to [3, 5, 6, 9, 10, 21, 22, 23] and the references therein.

Fractional differential equations have extensive applications in various fields of science and engineering. Many phenomena in viscoelasticity, electrochemistry, control theory, porous media, electromagnetism, and other fields, can be modeled by fractional differential equations. We refer the reader to [18, 24] and references therein for some applications. Fractional BVPs defined on intervals have been studied by many authors. Many results on the existence, uniqueness, multiplicity, and nonexistence of solutions for fractional differential equations subject to various boundary conditions (BCs) have been obtained; see for example [1, 2, 4, 7, 8, 11, 12, 13, 14, 15, 16, 17, 19, 20, 25, 26, 27].

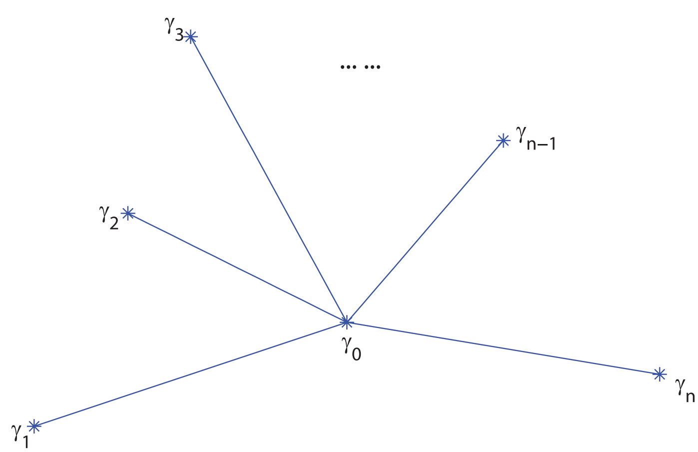

Figure 1: A star graph with n edges

To the best of our knowledge, no work has been done for fractional BVPs on graphs. This is in part due to the fact that many useful techniques for integer order differential equations fail to work for the corresponding fractional order problems. One example of this is the construction of Green's functions (see the discussion in Section **2** below). In this paper, we consider a fractional BVP on a star graph, which is a graph consisting of n edges with a common node γ_0 (see Figure **1**). For the purpose of simplification, we focus on a star graph consisting of three nodes and two edges, i.e., $G = V \cup E$ with $V = \{\gamma_0, \gamma_1, \gamma_2\}$ and $E = \{\overrightarrow{\gamma_1\gamma_0}, \overrightarrow{\gamma_2\gamma_0}\}$, where γ_0 is the junction node and $\overrightarrow{\gamma_i\gamma_0}$ is the edge connecting γ_i and γ_0 with length $l_i = |\overrightarrow{\gamma_i\gamma_0}|$, $i = 1, 2$. On each edge $\overrightarrow{\gamma_i\gamma_0}$, $i = 1, 2$, a local coordinate system with the origin at node γ_i and the coordinate $x \in (0, l_i)$ is introduced. We will study a BVP on G consisting of nonlinear fractional differential equations defined on $\overrightarrow{\gamma_i\gamma_0}$ by

$$-D^{\alpha}_{0+}\mathfrak{u}_i = \mathfrak{w}_i(x)\mathfrak{f}_i(x, \mathfrak{u}_i), \quad 0 < x < l_i, \; i = 1, 2, \tag{1.1}$$

and the Dirichlet BC defined at boundary nodes γ_1 and γ_2,

$$\mathfrak{u}_1(0) = \mathfrak{u}_2(0) = 0, \tag{1.2}$$

together with conditions of conjunctions at γ_0,

$$\mathfrak{u}_1(l_1) = \mathfrak{u}_2(l_2), \; D^{\beta}_{0+}\mathfrak{u}_1(l_1) + D^{\beta}_{0+}\mathfrak{u}_2(l_2) = 0, \tag{1.3}$$

where $D^{\alpha}_{0+}h$ is the α-th Riemann-Liouville fractional derivative of $h : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$D^{\alpha}_{0+}h(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}h(s)ds, \quad n = \lfloor \alpha \rfloor + 1, \tag{1.4}$$

provided the right-hand side exists, where Γ is the Gamma function. We assume the following conditions are satisfied:

(i) $1 < \alpha \le 2$ and $0 < \beta < \alpha$,
(ii) $\mathfrak{w}_i \in C[0, l_i]$ with $\mathfrak{w}_i(x) \not\equiv 0$ on $[0, l_i]$ and $\mathfrak{f}_i \in C([0, l_i] \times \mathbb{R}, \mathbb{R})$, $i = 1, 2$.

Remark **1.1**. When $\beta = 1$, BC (1.3) is the ordinary conjunction condition at γ_0.

It will be shown by a transformation that BVP (1.1)–(1.3) is equivalent to a BVP system defined on $[0, 1]$ with mixed BCs. Then two results on the existence and uniqueness of solutions will be obtained via fixed point theory. Our work fills the void in this area and can easily be extended to more general star graphs with n edges.

This paper is organized as follows: After this introduction, our main results are stated in Section **2**. Two examples are also given there. All the proofs are given in Section **3**, and the final section contains some observations and suggestions for future work.

2. Main results

We first convert BVP (1.1)–(1.3) to a BVP system defined on $[0, 1]$.

Theorem **2.1**. *BVP* (1.1)–(1.3) *is equivalent to a BVP system defined on* $[0,1]$ *by*

$$-D_{0+}^{\alpha}u_i = w_i(t)f_i(t,u_i), \quad 0<t<1, \tag{2.1}$$
$$u_1(0)=u_2(0)=0, \tag{2.2}$$
$$u_1(1)=u_2(1),\ l_1^{-\beta}D_{0+}^{\beta}u_1(1)+l_2^{-\beta}D_{0+}^{\beta}u_2(1)=0, \tag{2.3}$$

where $u_i(t)=\mathfrak{u}_i(l_it)$, $w_i(t)=l_i^{\alpha}\mathfrak{w}_i(l_it)$, *and* $f_i(t,z)=\mathfrak{f}_i(l_it,z)$, $i=1,2$.

An approach that is often used in studying BVPs on graphs is to convert the equation to a system of equations on an interval. But the fractional problem obtained involves a mixed boundary condition, which is quite unusual. Moreover, the Green function for such a problem is not available. The method we use below to construct our Green's function is new.

For $(t,s)\in[0,1]\times[0,1]$, define

$$G_{11}(t,s)=\begin{cases} t^{\alpha-1}g_{11}(s), & t\le s,\\ t^{\alpha-1}g_{11}(s)-(t-s)^{\alpha-1}/\Gamma(\alpha), & s\le t,\end{cases} \tag{2.4}$$
$$G_{12}(t,s)=t^{\alpha-1}g_{12}(s), \tag{2.5}$$
$$G_{21}(t,s)=t^{\alpha-1}g_{21}(s), \tag{2.6}$$
$$G_{22}(t,s)=\begin{cases} t^{\alpha-1}g_{22}(s), & t\le s,\\ t^{\alpha-1}g_{22}(s)-(t-s)^{\alpha-1}/\Gamma(\alpha), & s\le t,\end{cases} \tag{2.7}$$

where

$$g_{11}(s)=[l_2^{-\beta}(1-s)^{\alpha-1}-l_1^{-\beta}(1-s)^{\alpha-\beta-1}]/\tilde{\Delta}, \tag{2.8}$$
$$g_{12}(s)=[l_2^{-\beta}(1-s)^{\alpha-\beta-1}-l_2^{-\beta}(1-s)^{\alpha-1}]/\tilde{\Delta}, \tag{2.9}$$
$$g_{21}(s)=[l_1^{-\beta}(1-s)^{\alpha-\beta-1}-l_1^{-\beta}(1-s)^{\alpha-1}]/\tilde{\Delta}, \tag{2.10}$$
$$g_{22}(s)=[l_2^{-\beta}(1-s)^{\alpha-\beta-1}+l_1^{-\beta}(1-s)^{\alpha-1}]/\tilde{\Delta}, \tag{2.11}$$

with

$$\tilde{\Delta}=\Gamma(\alpha-\beta)\begin{vmatrix} 1 & -1\\ l_1^{-\beta}\Gamma(\alpha)/\Gamma(\alpha-\beta) & l_2^{-\beta}\Gamma(\alpha)/\Gamma(\alpha-\beta)\end{vmatrix}.$$

Remark **2.1**. By (2.4)–(2.11), for $i,j=1,2$ with $\beta\in(0,\alpha-1]$, $G_{ij}(t,s)$ is continuous on $[0,1]\times[0,1]$; if $\beta\in(\alpha-1,\alpha)$, $G_{ij}(t,s)$ becomes singular at $s=1$. For both cases, $G_{ij}(t,\cdot)\in L^1(0,1)$, $t\in[0,1]$.

Define

$$U = \max_{t\in[0,1]} \left\{ \int_0^1 |G_{ij}(t,s)w_j(s)|ds,\ i,j=1,2 \right\}. \tag{2.12}$$

The following are our main results.

THEOREM **2.2**. *Assume $\mathfrak{f}_i$ satisfies the Lipschitz condition in z,*

$$|\mathfrak{f}_i(x,z_1) - \mathfrak{f}_i(x,z_2)| \le K_i|z_1 - z_2| \text{ for } (x,z_1),(x,z_2) \in [0,l_i]\times\mathbb{R}, \tag{2.13}$$

with $K_i \in (0, 1/(4U))$, $i = 1,2$. Then BVP (1.1)–(1.3) has a unique solution. Furthermore, if $\mathfrak{f}_i(x,0) \equiv 0$ on $[0,l_i]$, $i = 1,2$, then BVP (1.1)–(1.3) has no nontrivial solutions.

THEOREM **2.3**. *Assume*

$$\lim_{|z|\to\infty} \max_{x\in[0,l_i]} \frac{\mathfrak{f}_i(x,z)}{|z|} = 0 \tag{2.14}$$

and $\mathfrak{f}_i(x,0) \not\equiv 0$ on $[0,l_i]$, $i = 1,2$. Then BVP (1.1)–(1.3) has at least one nontrivial solution.

To illustrate the application of our results, let us consider the following examples. We assume (i) and (ii) are satisfied with $\mathfrak{w}_i(x) \equiv 1$, $i = 1,2$, $l_1 = 1$, $l_2 = 2$, and U is defined by (2.12).

EXAMPLE **2.1**. Consider the BVP (1.1)–(1.3) with

$$\begin{cases} \mathfrak{f}_1(x,z) = p\tan^{-1} z + e^x, & (x,z) \in [0,l_1]\times\mathbb{R}, \\ \mathfrak{f}_2(x,z) = pz, & (x,z) \in [0,l_2]\times\mathbb{R}, \end{cases}$$

where $0 < p < 1/(4U)$. It is easy to see that $\mathfrak{f}_1$ and $\mathfrak{f}_2$ satisfy (2.12) with $k_1 = k_2 = p$. Then by Theorem **2.2**, BVP (1.1)–(1.3) has a unique solution.

EXAMPLE **2.2**. Consider the BVP (1.1)–(1.3) with

$$\begin{cases} \mathfrak{f}_1(x,z) = \tan^{-1} z + e^x, & (x,z) \in [0,l_1]\times\mathbb{R}, \\ \mathfrak{f}_2(x,z) = \sqrt[3]{z} + \cos x, & (x,z) \in [0,l_2]\times\mathbb{R}. \end{cases}$$

It is easy to check that the conditions of Theorem **2.3** are satisfied, so BVP (1.1)–(1.3) has at least one nontrivial solution.

3. Proofs of the main results

We first derive a lemma to prove Theorem **2.1**.

LEMMA **3.1**. *Let $\mathfrak{u}$ be a function defined on $[0,l]$ and $\alpha > 0$. Assume $D^{\alpha}_{0+}\mathfrak{u}$ exists on $(0,l]$. Let $x \in [0,l]$, $t = x/l \in [0,1]$, and $u(t) = \mathfrak{u}(lt)$. Then $(D^{\alpha}_{0+}\mathfrak{u})(x) = l^{-\alpha}(D^{\alpha}_{0+}u)(t)$.*

P r o o f. By (1.4), for $x \in (0,l]$, we have

$$\begin{aligned}(D^{\alpha}_{0+}\mathfrak{u})(x) =& \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-s)^{n-\alpha-1}\mathfrak{u}(s)ds\\ =& \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-s)^{n-\alpha-1}u(\check{s})ds,\end{aligned}$$

where $\check{s} = s/l$. Then by a change of variables,

$$(D^{\alpha}_{0+}\mathfrak{u})(x) = \frac{1}{l^{\alpha}\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\check{s})^{n-\alpha-1}u(\check{s})d\check{s} = l^{-\alpha}(D^{\alpha}_{0+}u)(t).$$

□

Theorem **2.1** is a direct application of Lemma **3.1**. We omit the proof.

Next, we use fixed point theory to study the existence of solutions of BVP (2.1)–(2.3). An operator associated to such a BVP will be constructed based on the following two lemmas. The reader is referred to Bai and Lü [4, Lemma 2.2] and [18, Property 2.1 and Lemma 2.3] for details of the proofs.

LEMMA **3.2**. *Assume that $\alpha \in (1,2]$ and $u \in C[0,1]$ has an α-th fractional derivative that belongs to $C[0,1]$. Then*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$$

for some $c_1, c_2 \in \mathbb{R}$, where I^{α}_{0+} is the α-th Riemann-Liouville integral of h defined by

$$I^{\alpha}_{0+}h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds. \tag{3.1}$$

LEMMA **3.3**. *Assume $\alpha > \beta \geq 0$. Then*

(a) $D^{\beta}_{0+}I^{\alpha}_{0+}u(t) = I^{\alpha-\beta}_{0+}u(t)$.
(b) $D^{\beta}_{0+}t^{\alpha-1} = \Gamma(\alpha)t^{\alpha-\beta-1}/\Gamma(\alpha-\beta)$.

In the sequel, let $X = C([0,1],\mathbb{R}^2)$ and $\mathbf{0}$ be the zero of X. For $u = (u_1, u_2)^T \in X$, let $\|u\| = \max_{t\in[0,1]}(|u_1(t)| + |u_2(t)|)$. Then $(X, \|\cdot\|)$ is a Banach space. Define $T: X \to X$ by

$$Tu = ((T_1u), (T_2u))^T \tag{3.2}$$

with

$$(T_iu)(t) = \int_0^1 G_{i1}(t,s)w_1(s)f_1(s,u_1(s))ds + \int_0^1 G_{i2}(t,s)w_2(s)f_2(s,u_2(s))ds, \quad t \in [0,1],\ i = 1,2. \tag{3.3}$$

It is easy to see that T is a completely continuous operator. Furthermore, we can prove the following lemma.

LEMMA **3.4**. *BVP* (2.1)–(2.3) *has a solution if and only if T has a fixed point in X.*

P r o o f. We first consider a linear BVP system consisting of

$$-D_{0+}^{\alpha}u_i = h_i(t), \quad 0 < t < 1,$$

and BCs (2.2), (2.3), where $h_i(t) \in C[0,1]$, $i = 1,2$. By Lemma **3.2**, we have

$$u_i = -I_{0+}^{\alpha}h_i + c_1^{[i]}t^{\alpha-1} + c_2^{[i]}t^{\alpha-2}, \quad i = 1,2,$$

where $c_j^{[i]}$, $i,j = 1,2$, are some constants. BC (2.2) implies $c_2^{[1]} = c_2^{[2]} = 0$. Hence

$$u_i = -I_{0+}^{\alpha}h_i + c_1^{[i]}t^{\alpha-1}, \quad i = 1,2. \tag{3.4}$$

By Lemma **3.3**,

$$D_{0+}^{\beta}u_i = -I_{0+}^{\alpha-\beta}h_i + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}c_1^{[i]}t^{\alpha-\beta-1}, \quad i = 1,2.$$

Then BC (2.3) implies that $c_1^{[1]}$ and $c_1^{[2]}$ must satisfy

$$\begin{pmatrix} 1 & -1 \\ l_1^{-\beta}\Gamma(\alpha)/\Gamma(\alpha-\beta) & l_2^{-\beta}\Gamma(\alpha)/\Gamma(\alpha-\beta) \end{pmatrix} \begin{pmatrix} c_1^{[1]} \\ c_1^{[2]} \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \tag{3.5}$$

where

$$H_1 = (I_{0+}^{\alpha}h_1)(1) - (I_{0+}^{\alpha}h_2)(1) \quad \text{and} \quad H_2 = (I_{0+}^{\alpha-\beta}h_1)(1)/l_1^{\beta} + (I_{0+}^{\alpha-\beta}h_2)(1)/l_2^{\beta}.$$

Clearly,

$$\begin{vmatrix} 1 & -1 \\ l_1^{-\beta}\Gamma(\alpha)/\Gamma(\alpha-\beta) & l_2^{-\beta}\Gamma(\alpha)/\Gamma(\alpha-\beta) \end{vmatrix} > 0.$$

Hence, we have

$$c_1^{[1]} = \int_0^1 g_{11}(s)h_1(s)ds + \int_0^1 g_{12}(s)h_2(s)ds, \tag{3.6}$$

$$c_1^{[2]} = \int_0^1 g_{21}(s)h_1(s)ds + \int_0^1 g_{22}h_2(s)ds, \tag{3.7}$$

where g_{ij} are defined by (2.8)–(2.11), $i, j = 1, 2$. Substituting (3.6) and (3.7) into Eq. (3.4) and using (3.1), we see that

$$u_i(t) = \int_0^1 G_{i1}(t,s)h_1(s)ds + \int_0^1 G_{i2}(t,s)h_2(s)ds, \quad i, j = 1, 2, \tag{3.8}$$

where G_{ij} are defined by (2.4)–(2.7). The conclusion then follows immediately by letting $h_i(t) = w_i(t)f_i(t, u_i(t))$, $i = 1, 2$, in (3.8). □

Remark **3.1**. The idea of Lemma **3.4** can also be applied to the problems defined on star graphs G consisting of $n+1$ nodes and n edges, $n > 2$. By Lemma **3.1**, an equivalent n dimensional BVP system defined on $[0, 1]$ can be derived and the associated operator can be found by a similar idea to Lemma **3.4**. The key step is to set up an n dimensional linear system analogous to (3.5) based on the conjunction conditions at γ_0 and then solve for the constants $c_1^{[i]}$, $i = 1, \dots, n$.

Proof. (Proof of Theorem **2.2**.) We only need to prove that BVP (2.1)–(2.3) has a unique solution. Note that (2.13) implies that for $i = 1, 2$,

$$|f_i(t, z_1) - f_i(t, z_2)| \le K_i|z_1 - z_2| \text{ for } (t, z_1), (t, z_2) \in [0, 1] \times \mathbb{R}.$$

By (3.3), for any $u^{[1]}, u^{[2]} \in X$ and $i = 1, 2$,

$$\begin{aligned} &|(T_i u^{[1]})(t) - (T_i u^{[2]})(t)| \\ &\qquad \le \int_0^1 |G_{i1}(t,s)w_1(s)||f_1(s, u_1^{[1]}(s)) - f_1(s, u_1^{[2]}(s))|ds \\ &\qquad\quad + \int_0^1 |G_{i2}(t,s)w_2(s)||f_2(s, u_2^{[1]}(s)) - f_2(s, u_2^{[2]}(s))|ds \\ &\qquad \le \int_0^1 |G_{i1}(t,s)w_1(s)|K_1|u_1^{[1]}(s) - u_1^{[2]}(s)|ds \\ &\qquad\quad + \int_0^1 |G_{i2}(t,s)w_2(s)|K_2|u_2^{[1]}(s) - u_2^{[2]}(s)|ds. \end{aligned}$$

Hence, for $t \in [0,1]$ and $i = 1, 2$,

$$|(T_i u^{[1]})(t) - (T_i u^{[2]})(t)| \leq U(K_1 + K_2)\|u^{[1]} - u^{[2]}\|,$$

where U is defined by (2.12). Therefore,

$$\begin{aligned}\|Tu^{[1]} - Tu^{[2]}\| = \max_{t\in[0,1]} (&|(T_1 u^{[1]})(t) - (T_1 u^{[2]})(t)| \\ &+ |(T_2)u^{[1]}(t) - (T_2 u^{[2]})(t)|) \\ \leq\ & 2U(K_1 + K_2)\|u^{[1]} - u^{[2]}\|.\end{aligned}$$

Since $2U(K_1 + K_2) < 1$, T is a contraction operator. Therefore, BVP (2.1)–(2.3) has a unique solution, and so BVP (1.1)–(1.3) has a unique solution.

If in addition, $\mathfrak{f}_i(x, 0) \equiv 0$ on $[0, l_i]$, $i = 1, 2$, then obviously $u \equiv \mathbf{0}$ is a solution of BVP (2.1)–(2.3). By the uniqueness of solutions, BVP (2.1)–(2.3) has no nontrivial solutions. So is BVP (1.1)–(1.3). □

P r o o f. (P r o o f o f T h e o r e m **2.3**.) Let $k = 1/(4U)$, where U is defined by (2.12). For $i = 1, 2$, (2.14) implies

$$\lim_{|z|\to\infty} \max_{t\in[0,1]} \frac{f_i(t, z)}{|z|} = 0.$$

Hence there exists $M_1 > 0$ such that $|f_i(t, z)| \leq k|z|$ for any $t \in [0, 1]$ and z with $|z| \geq M_1$. The continuity of f_i implies there exists $F > 0$ such that $|f_i(t, z)| \leq F$ for any (t, z) on $[0, 1] \times [-M_1, M_1]$. Let $M_2 = \max\{M_1, F/k\}$. Then

$$|f_i(t, z)| \leq kM_2 \quad \text{for any } (t, z) \text{ on } [0, 1] \times [-M_2, M_2],\ i = 1, 2. \tag{3.9}$$

Let $\Omega = \{u \in X \mid \|u\| \leq M_2\}$. For any $u \in \Omega$, $|u_i(t)| \leq M_2$ on $[0, 1]$, $i = 1, 2$. By (3.3) and (3.9), for $t \in [0, 1]$ and $i = 1, 2$,

$$\begin{aligned}|(T_i u)(t)| \leq & \int_0^1 |G_{i1}(t, s)w_1(s)||f_1(s, u_1(s))|ds \\ & + \int_0^1 |G_{i2}(t, s)w_2(s)||f_2(s, u_2(s))|ds \\ \leq\ & kM_2 \left(\int_0^1 |G_{i1}(t, s)w_1(s)|ds + \int_0^1 |G_{i2}(t, s)w_2(s)|ds \right) \\ \leq\ & M_2/2.\end{aligned}$$

Hence, $\|Tu\| \leq M_2$, i.e., $T(\Omega) \subset \Omega$. By the Schauder fixed point theorem, T has at least one fixed point in Ω. Clearly $u \equiv \mathbf{0}$ is not a fixed point. Therefore, BVP (2.1)–(2.3) has at least one nontrivial solution and so BVP (1.1)–(1.3) has at least one nontrivial solution. □

4. Concluding remarks and suggestions for further research

As we pointed out in Introduction, we focused on a star graph containing three nodes and two edges. This was so that we could demonstrate the characteristics of our approach while maintaining some simplicity in the presentation. As noted in Remark **3.1** above, a key lemma in our proof, namely Lemma **3.4**, can be extended to problems defined on star graphs having $n+1$ nodes and n edges with $n > 2$. It would of course interesting to see if our approach can be applied to graphs with more general structures.

Our work here was not motivated by a specific applied problem such as those described in the very nice survey paper of Kuchment [21], but it is hoped that our approach to treating fractional boundary value problems on graphs will be of use to studying such problems.

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