12-15-2000

Fractional derivatives

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FRACTIONAL DERIVATIVES

By

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A Thesis

Submitted in partial fulfillment of the requirements of the
Master of Arts Degree
Of
The Graduate School
At
Rowan University
December 15, 2000

Approved by ______________________________
Professor

Date Approved Dec. 15, 2000

ABSTRACT

John M. Beach

Fractional Derivatives

Year: 2000

Advisor: Dr. Marcus W. Wright

Mathematics Program

In this thesis, the reader will not find a study of any kind; there is no methodology, questionnaire, interview, test, or data analysis. This thesis is simply a research paper on fractional derivatives, a topic that I have found to be fascinating. The reader should be delighted by a short history of the topic in Chapter 1, where he/she will read about the contributions made by some of the great mathematicians from the last three centuries.

In Chapter 2 the reader will find and intuitive approach for finding the general fractional derivative for functions such as e^{ax}, x^p, and $f(x)$. Other topics in Chapter 2 include branch lines and the Weyl Transform. All of the work preformed by an intuitive approach is backed up by a rigorous approach using Complex Analysis in Chapter 3. In Chapter 4 the reader will find an excellent application of fractional derivatives in solving the tautochrone problem.

No paper on fractional derivatives could be complete with out a Chapter (5) on Oliver Heaviside. Heaviside's thoughts on rigorous formalism and his use of non-logical mathematics should delight the reader.

Lastly, the reader should enjoy my final thoughts on this topic as well as Heavisides' thoughts.

MINI-ABSTRACT

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Acknowledgment

The author wishes to express his indebtedness to the following. Dr. Marcus W. Wright of Rowan University, Mathematics Department for his guidance and encouragement throughout my years at the University, and to Professor Thomas J. Osler, from the same institution, for his valuable references and lectors, without which this paper would not be possible. Also, I express my indebtedness to my wife Lori for her patience and understanding do to the lack of time that we had together. Also, to my family and friends for their understanding and encouragement, and to my mother for her countless prayers.

Lastly, I wish to express my indebtedness to my two brothers Frank and Robert for their endless pushing, (or should I say nagging) and encouragement. With special thanks to Frank for his excellent proof-reading, and to Robert for talking me into going back to school six and a half years ago, for which none of this would be possible.

CONTENTS

Chapter 1

A Short History of Fractional Derivatives

Leibniz 1690's

The origin of fractional derivatives is not certain at this time, but we do know that Leibniz, the inventor of the notation $d^n y / dx^n$, had "toyed" with the idea in the 1600's. In 1695 L'Hopital asked Leibniz: "What if n be ½?" Surprisingly Leibniz [1] replied: "...You can see by that, sir, that one can express by an infinite series a quantity such as $d^{1/2}xy$ or $d^{1:2}xy$. Although infinite series and geometry are distant relations, infinite series admits only the use of exponents that are positive and negative integers, and does not, as yet, know the use of fractional exponents..." As with most great mathematicians, Leibniz had a unique insight into the unknown. He stumbled onto fractional derivatives realizing that one-day great things will come from his work. What they would be, he had no idea. In the same letter he continued: "...Thus it follows that $d^{1/2}x$ will be equal to $x\sqrt{dx : x}$. This is an apparent paradox from which, one day, useful consequences will be drawn..." Leibniz insight did not stop there. Three years latter in a letter to John Wallis, he discussed ways of using fractional derivatives in Wallis's infinite product for $\frac{1}{2}\pi$. He states [2]: "...Differential calculus might have been used to achieve this result..." It should be evident that Leibniz did not have just a passing thought on fractional derivatives, he must have spent a considerable amount of time on the topic. I wish I were a fly on the wall in the 1600's.

Euler 1738

Euler, another great mathematician, toyed with the idea of fractional derivatives. 43 years after Leibniz went pubic with his controversial ideas of fractional derivatives, Euler stated in his 1738 dissertation [3]: "…When n is a positive integer, and if p should be a function of x, the ratio $d^n p$ to dx^n can always be expressed algebraically, so that if n = 2 and $p = x^3$, then d^2x^3 *to* dx^2 is 6x to 1. Now it is asked what kind of ratio can then be made if n be a fraction. The difficulty in this case can easily be understood. For if n is a positive integer d^n can be found by continued differentiation. Such a way, however, is not evident if n is a fraction. But yet with the help of interpolation which I have already explained in this dissertation, one may be able to expedite the matter…"

Searching through several books on this topic I only found one "hit" in 80 years after Euler's dissertation, that "hit" was Laplace. In 1812 Laplace mentioned, in passing, fractional derivatives by means of integrals. If he was around today I am sure he would be sorry he did not do more on the subject.

Lacroix 1819

In 1819 Lacroix wrote a 700-page textbook on differential and integral calculus. He stumbled over fractional derivatives in a two-page exercise; he develops the *n*th derivative and then generalizes it with the gamma functions. He finishes the exercise with an example for when y = x and $n = \frac{1}{2}$; he obtained:

$$\frac{d^{1/2}y}{dx} = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

It appears to me that Lacroix "missed the boat" on fractional derivatives. This will become evident to the reader in Chapter 2, equation a_8.

Fourier 1822

Three years later Fourier wrote about derivatives of arbitrary order where he generalized his formula using u as an arbitrary number. He obtained:

$$\frac{d^u}{dx^u} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} p^u \cos\left[p(x-\alpha) + \tfrac{1}{2} u\pi\right] dp$$

He stated [6]: "...The number u that appears above will be regarded as any quantity whatsoever, positive or negative..." Too bad Fourier did not go farther with this topic.

Abel 1823

Up to this point in time mathematicians only "played" with the notion of fractional derivatives. One year after Fourier, Abel [7] took the proverbial ball and ran with it. While Able was "toying" with the tautochrone problem he stumbled over the solution by using fractional calculus. Examples of the tautochrone problem will be shown later. Without going too far into the solution, Abel's general integral equation for k is given as follows:

$$k = \int_0^{\infty} (x-t)^{\frac{1}{2}} f(t)dt$$

Where k is a known constant for the amount of time it takes for a frictionless mass to slide down a curve no matter where the mass starts. The function f is unknown and will be determined at a latter time.

Able "played" with general integral equations until he came up with the following:

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} k = \sqrt{\pi} f(x) \qquad \text{(Where } k \text{ is a known constant)}$$

Abel used Fourier's integral formulas to solve his problem but never gave him credit for the solution.

Liouville 1832

Nine years after Abel's solution, in 1832, the famous mathematician Liouville published three memoirs, which were the fruit of the first major study in fractional derivatives. Shortly after his three memoirs, Liouville published several papers on theoretical applications using fractional derivatives in the solutions. Liouville begun with a well known formula of his time:

$$D^m e^{ax} = a^m e^{ax}$$

He then let v be a derivative with arbitrary order, which yielded:

$$D^v e^{ax} = a^v e^{ax}$$

He "played" with it in a intuitive way with derivatives of arbitrary order and expanded the formula in a series until he came up with:

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}, \qquad Re\ a_n > 0 \qquad \textbf{(a)}$$

Which yielded:

$$D^v f(x) = \sum_{n=0}^{\infty} c_n a_n^v e^{anx}$$

The above formula is sometimes known as Liouville's first formula of fractional derivatives, which is an intuitive approach of arbitrary order v, where Liouville allowed v to be any number; rational, irrational, or complex. It should be easy to see that Liouville's first formula is applicable to functions only in the form of *(a)*.

Liouville may or may not have been aware of the narrowness of his first formula for fractional derivatives, but he came up with his second formula of fractional derivatives. He started with a definite integral:

$$I = \int_0^{\infty} u^{a-1} e^{-xu} du \quad , \qquad a > 0, \quad x > 0$$

He "played" with the formula by changing variables and operating on both sides with D^v to obtain his second formula of fractional derivatives:

$$D^v x^{-a} = \frac{(-1)^v \Gamma(a+v)}{\Gamma(a)} x^{-a-v} \quad , a > 0 \qquad \textbf{(b)}$$

Where v is any number rational, irrational, or complex.

Although Liouville was the first to try solving fractional differential equations, he was not totally correct. He realized that his first and second formulas for fractional derivatives needed too narrow restrictions to be of much use. His first formula was only good for the class of (a), and his second formula was only good for functions in the form x^{-a} with $a > 0$. It is clear that Liouville was aware of this fact since in one of his memoirs of 1834 [5] he says: "...The ordinary differential equation $d^n y / dx^n = 0$ has the complementary solution $y_c = c_0 + c_1 x + c_2 x^2 + \ldots + c_{n-1} x^{n-1}$. Thus $d^u y / dx^u = 0$ (u arbitrary) should have a corresponding complementary solution..." While Liouville did come up with a corresponding complementary solution it became the center of controversy during his time. One would wish that he had gone further in developing this topic.

The Fight of the Decade

From 1833 to 1848 several mathematicians ended up fighting over the work of Lacroix, Able, and Liouville. In 1833 Peacock supported Lacroix's formula, while holding Liouville's formulas as being useless except for a few special cases. Peacock made several errors while trying to support Lacroix's formula; one of his biggest errors was misapplication of symbolic operations, where he believed that the principles of symbolic algebra would hold true for derivatives.

On the other hand Kelland supported Liouville on two separate occasions in 1839 and the other time in 1846 when he believed that Liouville's second formula had useful implications in the form of x^{-a}.

In 1840 De Morgan [6] writes (referring to Lacroix formula and Liouville's second formula): "...Both these systems may very possibly be part of a more general system, but at present I incline to the conclusion that neither system has any claim to be considered as given the form $D^n x^m$, though either may be a form..." Even De Morgan, one of the great mathematics of all times, could not make up his mind on this matter. In 1848, William Center could not make up his mind either. He stated [7]: "...according to Liouville's system, by letting $a = 0$ the fractional derivative of unity equals zero because $\Gamma(0) = \infty$...The whole question is plainly reduced to what is $d^u x^0 / dx^u$. For when this is determined we shall determine at the same time which is the correct system..."

Well, who was right? It turns out that De Morgan was correct for both Lacroix formula and Liouville's second formula were incorporated into a more general formula years later.

Riemann (late 1800's)

Exactly when Riemann worked on fractional derivatives no one knows, for he never publicized any of it. But we do know he did his work in his student years. Riemann tried to find the general solution by way of the Taylor series and letting Ψ(x) be the complementary function, which yielded:

$$D^{-v} f(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} f(t)dt + \Psi(x) \qquad \textbf{(c)}$$

No one is sure that Riemann knew exactly what the outcome of a complementary function would be, for he used it to provide a "measure of the deviation". In 1880 Cayley [8] stated: "...The greatest difficulty in Riemann's theory, it appears to me, is the question of the meaning of a complementary function containing an infinity of arbitrary constants...Any satisfactory definition of a fractional operation will demand that this difficulty be removed..." Later in his paper, Cayley says: "...Riemann was hopelessly entangled in his version of a complementary function..." All too many times, when we become to close to a project that we are working on, we can not see the trees through the proverbial forest. It appears to me that Riemann had this same problem. Riemann did little more with this topic, but we will see that he had tremendous insight, and several mathematicians built on his work.

Laurent 1884

Two mathematicians, Sonin and Letnikov, developed the prelude to the idea of fractional derivatives for modern mathematicians. In 1869 Sonin wrote a paper, "On Differentiation With Arbitrary Index", and Letnikov wrote four papers between 1868 and 1872 on the same topic. Both mathematicians started their work with Cauchy's integral formula:

$$D^n f(z) = \frac{n!}{2\pi i}\int_c \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \qquad \textbf{(d)}$$

Where c represents a closed contour going around once counter clockwise. Sonin and Letnikov were off to a great start since it was permitted to generalize $n!$. Both knew about the gamma function and how $v! = \Gamma(v + 1)$ when $v!$ takes on arbitrary values of integers. They knew when n was an integer they would obtain a simple pole in the contour of the close circuit. They saw when n was not an integer they would no longer have a simply pole but a branch cut. Sonin and Letnikov realized the problem but did not provide a solution.

Unfortunately for Sonin and Letnikov for, 12 years later, in 1884, Laurent solved the problem. Laurent, as well, started with Cauchy's integral formula **(d).** He used the rules of transformation and his contour was an open path on a Riemann surface. He produced his definition for differentiation for arbitrary order:

$${}_cD_x^{-v} f(x) = \frac{1}{\Gamma(v)}\int_c^x (x-t)^{v-1} f(t)dt \quad , \qquad \operatorname{Re} v > 0 \qquad \textbf{(e)}$$

Do you notice what happens if we let $x > c$ in Laurent's definition **(e)**? You should see that it is Riemann's definition **(c)** without his complementary function $\Psi(x)$. It is important to note that when $c = 0$ Laurent obtained:

$$ {}_0D_x^{-v} f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t)dt \quad , \qquad \text{Re } v > 0 $$

This version is the most commonly used, and is named the Riemann-Liouville fractional integral. I believe that it should be named the Liouville-Riemann fractional integral, since Liouville tried to solve the problem first. In any event he finally received recognition for his work. I wish he were around today to witness the fruits of his labor.

Heaviside 1892

Oliver Heaviside, a genius in his time, has become one of my "heroes," although he was an untrained scientist, (as stated by Miller and Ross, [10], pp. 13), and not a mathematician. I look at things the same way as he did. I prefer to use an intuitive approach when looking at problems. This was much more common in previous centuries than now. In 1892 he published several papers on linear functional operators, where his unorthodox methods led to solving certain engineering problems such as, transmission of electrical currents in cables, temperature distribution, and the submarine cable equation. His brilliant methods, solutions, and applications have been collected and named, "Heaviside operational calculus." But, back in his time, his work was looked at with suspicion and distrust. He became a laughing stock of the mathematics community since he was unable to back up his work with rigorous proofs. I can only thank God for a mathematician by the name of Bromwich. In 1919 Bromwich set out to prove all of Heaviside's work, which he did by rigorous proofs.

1892 to 1974

It is surprising that 82 years have gone by and only a relatively few research papers have been written on the topic of fractional derivatives, especially since there has been an explosion of new mathematicians during this time. Some of the few "greats"' are: Al-Bassam, Davis Erdelyi, Hardy, Kobler, Littewood, Love, Riesz, Samko, Sneddon, Weyl, Zygmund, and our own Dr. Thomas J. Osler. With all these new mathematicians coming on the scene, one would think there would be hundreds if not thousand of research papers. Even Davis [9] in 1936 said: "...The period of the formal development of operational methods may be regarded as having ended by 1900. The theory of integral equations was just beginning to stir the imagination of mathematicians and to reveal the possibilities of operational methods..." It seems to me, not a whole lot of imaginations were being stirred in 82 years. But then came the year, 1974.

The Great Explosion of 1974

1974, This was the year that research into fractional derivatives really exploded. The very first international conferences on fractional calculus happened in 1974. It was held at the University of New Haven. Some of the above mentioned mathematicians were in attendance as well as Askey, Mikolas and many others, including our own Dr. Thomas J. Osler. The above heading did say "great explosion." The 1974 conference really stirred the imagination of many of the above mentioned. In just a little over five years there were more papers written on fractional derivatives then there even was since the beginning of mathematical time, about 400 total.

Then came the 1980's. The second international conference on fractional calculus took place ten years latter in 1984. It was held at the University of Strathclyde, Glasgow, Scotland. It seems that mathematicians from all over the world had jumped onto the proverbial bandwagon. Mathematicians from Japan, Soviet Union, England, India, Canada, Venezuela, Scotland, and a host of smaller nations all have written on the topic. Some of these mathematicians that wrote on the fractional calculus include: Saigo (1980), Owa (1990), and Nishimoto (1984, 1987, 1989, 1991) who wrote a four-volume set on applications The three mathematicians mentioned above are from Japan. In 1987 Marichev and Kilbas, from the Soviet Union, wrote an encyclopedia on the topic, along with applications. Rauna and Sexena from India wrote several papers in the 1980's. Srivastava from Canada, Kalla from Venezuela, and McBride from Scotland all made it to the "top" from their work on fractional derivatives. Even our own Dr. Thomas Osler published or co-published 10 papers on the topic in the 1980's and 90's.

One would think that with the thousands of mathematicians in the world today there would be countless volumes of published works on this topic. Unfortunately, the fact is most mathematicians have no idea of the opportunities and applications of fractional calculus. Many would not even know where to start if given a simple problem. Even worse is the fact many have only heard of fractional derivatives in passing and some not at all.

I would like to end this short history with a quote from Miller and Ross, [10]. They stated: "…The fractional calculus finds use in many fields of science and engineering, including fluid flow, rheology, diffusive transport akin to diffusion, electrical networks, electromagnetic theory, and probability. Some papers by P. C. Phillips [1989, 1990] have used the fractional calculus in statistics. R. L. Bagley [1990]; Bagley and Torvik [1986] have found uses for the fractional calculus in viscoelasticity and the electrochemistry of corrosion. It seems that hardly a field of science or engineering has remained untouched by this topic. Yet even though the subject is old, it is rarely included in today's curricula. Possibly, this is because many mathematicians are unfamiliar with its uses…"

Just in case the reader believes the rest of this paper will be a waste of time to read since the topic is not included in today's curricula. I will "tease you into reading the rest of the paper by telling you now that I will show the solution to the tautochrone problem through the use of fractional derivatives as an application problem in Chapter 4 of this paper.

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Chapter 2

An Intuitive Approach

I would like to start at the "heart" of this paper by looking at a few well-known functions, and try to find various derivatives by means of a intuitive approach. I will be making use of the usual notation for derivatives as follows:

$$\frac{df(x)}{dx} = D^1 f(x), \qquad \frac{df(x)}{dx^2} = D^2 f(x), \qquad \frac{df(x)}{dx^{1/2}} = D^{1/2} f(x), \ldots$$

Derivatives of e^{ax}

Let us begin to look at the derivatives in the form of e^{ax}:

$$\begin{array}{lll}
D^0\, e^x = e^x & D^0\, e^{2x} = e^{2x} & D^0\, e^{3x} = e^{3x} \\
D^1\, e^x = e^x & D^1\, e^{2x} = 2e^{2x} & D^1\, e^{3x} = 3e^{3x} \\
D^2\, e^x = e^x & D^2\, e^{2x} = 4e^{2x} = 2^2 e^{2x} & D^2\, e^{3x} = 9e^{3x} = 3^2 e^{3x} \\
D^3\, e^x = e^x & D^3\, e^{2x} = 8e^{2x} = 2^3 e^{2x} & D^3\, e^{3x} = 27e^{3x} = 3^3 e^{3x} \\
\vdots & \vdots & \vdots \\
D^n\, e^x = e^x & D^n\, e^{2x} = 2^n e^{2x} & D^n\, e^{3x} = 3^n e^{3x}
\end{array}$$

May I be so bold as to generalize the derivatives for e^{ax}? Well, I can if I give the stipulation that $n \geq 0$ and n must be an integer:

$$D^n\, e^{ax} = a^n e^{ax} \qquad (a_1)$$

Let us try to apply (a_1) to integers when $n \leq 0$. We will obtain the following:

$$\begin{array}{lll}
D^{-1}\, e^x = e^x & D^{-1}\, e^{2x} = 1/2e^{2x} = 2^{-1} e^{2x} & D^{-1}\, e^{3x} = 1/3e^{3x} = 3^{-1} e^{3x} \\
D^{-2}\, e^x = e^x & D^{-2}\, e^{2x} = 1/4e^{2x} = 2^{-2} e^{2x} & D^{-2}\, e^{3x} = 1/9e^{3x} = 3^{-2} e^{3x} \\
D^{-3}\, e^x = e^x & D^{-3}\, e^{2x} = 1/8e^{2x} = 2^{-3} e^{2x} & D^{-3}\, e^{3x} = 1/27e^{3x} = 3^{-3} e^{3x} \\
\vdots & \vdots & \vdots \\
D^{-n}\, e^x = e^x & D^{-n}\, e^{2x} = 2^{-n} e^{2x} & D^{-n}\, e^{3x} = 3^{-n} e^{3x}
\end{array}$$

Do you notice anything about the above results do they look familiar? They should, they are the indefinite integral as listed below:

$$D^{-1}e^{x} = \int e^{x}dx \qquad D^{-1}e^{2x} = \int e^{2x}dx \qquad D^{-1}e^{3x} = \int e^{3x}dx$$

$$D^{-2}e^{x} = \iint e^{x}dxdx \qquad D^{-2}e^{2x} = \iint e^{2x}dxdx \qquad D^{-2}e^{3x} = \iint e^{3x}dxdx$$

$$D^{-3}e^{x} = \iiint e^{x}dxdxdx \qquad D^{-3}e^{2x} = \iiint e^{2x}dxdxdx \qquad D^{-3}e^{3x} = \iiint e^{3x}dxdxdx$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$D^{-n}e^{x} = \int (\text{n - times})\ e^{x}dx(\text{n - times})$$

$$D^{-n}e^{2x} = \int (\text{n - times})\ e^{2x}dx(\text{n - times})$$

$$D^{-n}e^{3x} = \int (\text{n - times})\ e^{3x}dx(\text{n - times})$$

May I boldly go where no graduate student have gone before and say that (a_1) is valid for all n positive or negative as long as they are integers. Being as bold as I am, it is not unreasonable to say that if $n \geq 0$ and n is an integer, then (a_1) will become the derivative formula for e^{ax}, and if $n < 0$ and n is an integer, then (a_1) will become the indefinite integral formula for e^{ax}.

Now comes the first "meaty" question. Does (a_1) hold true if n is not an integer, better yet what if $n = ½$? Will $D^{1/2}e^{ax} = a^{1/2}e^{ax}$ hold true, will the following be true?

$$D^{1/2}\, e^{x} = e^{x} \quad (a_2) \qquad\qquad D^{1/2}\, e^{2x} = 2^{1/2}e^{2x} \quad (a_3)$$

We know that the rules of calculus tell us that $D^n\, D^m\, y = D^{n+m}\, y$ let's try it out on the $D^{1/2}e^{x}$ and see if my bold assumptions are true. We know $D^{1}e^{ax} = ae^{ax}$, so let's see what happens.

$$D^{1/2}D^{1/2}e^{ax} = D^{1/2}\, a^{1/2}\, e^{ax} = a^{1/2}\, a^{1/2}\, e^{ax} = ae^{ax}$$

So far my bold assumption is holding true, (a_1) appears to be a good candidate.

I will now try to derive a second formula for the fractional derivative of e^x. We know that we can expand e^x in a Taylor Series as follows:

$$e^x = \sum_{k=o}^{\infty} \frac{1}{k!} x^k$$

Now making use of the gamma function to find the derivative of the power series above, will yield:

$$D^n e^x = \sum_{k=0}^{\infty} \frac{x^{k-n}}{\Gamma(k-n+1)} \qquad \textbf{(a}_4\textbf{)}$$

Now, let's try our new formula for a few derivatives and see what happens.

$$D^1 e^x = \sum_{k=0}^{\infty} \frac{x^{k-1}}{\Gamma(k-1+1)} = e^x$$
$$D^2 e^x = \sum_{k=0}^{\infty} \frac{x^{k-2}}{\Gamma(k-2+1)} = e^x$$
$$D^3 e^x = \sum_{k=0}^{\infty} \frac{x^{k-3}}{\Gamma(k-3+1)} = e^x$$

(The reader can verify that each of these summations reduce to the usual power series for e^x as given above.) Since it appears that the formula ***(a₄)*** is another good candidate for the derivative of e^x, let's see if it will work for fractional derivatives.

$$D^{1/2} e^x = \sum_{k=0}^{\infty} \frac{x^{k-1/2}}{\Gamma(k-1/2+1)} \qquad \textbf{(a}_5\textbf{)}$$

As we can see we have a problem, ***(a₂)*** does not match ***(a₅)***.

$$e^x \neq \sum_{k=0}^{\infty} \frac{x^{k-1/2}}{\Gamma(k-1/2+1)}$$

Just in case, in some way e^x was imbedded in ***(a₅)*** I consulted the computer program Mathematica, which gave a remarkable result:

$$D^{1/2}e^x = \frac{e^x x^{6/5} - \dfrac{e^x x^{6/5}\Gamma[-6/5, x]}{\Gamma[-6/5]}}{\sqrt{x}}$$

(I.e., the sum of the previous infinite series.)

Unfortunately, formula (a_2) does not equal (a_5), so there is no need for us see what happens to (a_3). Regrettably, this means that formula (a_1) or the formula (a_4) is not valid if n is a fraction. What went wrong? Which formula is invalid? I will unravel the secret latter, but for now let's look at the derivatives for x^P.

Derivatives of x^p

Let us now look at some derivatives in the form of x^p, using the normal notation.

$D^0 x = x$	$D^0 x^2 = x^2$	$D^0 x^3 = x^3$	$D^0 x^4 = x^4$	$D^0 x^p = x^p$
$D^1 x = 1$	$D^1 x^2 = 2x$	$D^1 x^3 = 3x^2$	$D^1 x^4 = 4x^3$	$D^1 x^p = px^{p-1}$
$D^2 x = 0$	$D^2 x^2 = 4$	$D^2 x^3 = 6x$	$D^2 x^4 = 12x$	$D^2 x^p = p(p-1)x^{p-2}$
$D^3 x = 0$	$D^3 x^2 = 0$	$D^3 x^3 = 6$	$D^3 x^4 = 24x$	$D^3 x^p = p(p-1)(p-2)x^{p-3}$
$D^4 x = 0$	$D^4 x^2 = 0$	$D^4 x^3 = 0$	$D^4 x^4 = 24$	$D^4 x^p = p(p-1)(p-1)(p-3)x^{p-4}$

It is not take a leap of faith to see the pattern that is emerging.

$$D^n x^p = p(p-1)(p-2)(p-3)\ldots(p-n+1)x^{p-n}$$

Now, if we use a little "trick" from Dr. Osler and multiply numerator and denominator of the right hand side of the above formula by (p – n)! we obtain:

$$D^n x^p = \frac{p!}{(p-n)!}x^{p-n} \qquad (a_6)$$

Again, I will be so bold as to say the above, (a_6) is the general formula for derivatives in the form x^p, if I give the stipulation that $n \geq 0$ and n must be an integer. As a matter of fact, it is not difficult to see that (a_6) is the general formula for the n^{th} derivative of the form x^p, if $n > 0$ and n is an integer.

Let us try to apply (a_6) to integers when $n < 0$. We will obtain the following:

$D^{-1} x = x^2/2$	$D^{-1} x^2 = x^3/3$	$D^{-1} x^3 = x^4/4$	$D^{-1} x^5 = x^6/6$
$D^{-2} x = x^3/6$	$D^{-2} x^2 = x^4/12$	$D^{-2} x^3 = x^5/20$	$D^{-2} x^5 = x^7/42$
$D^{-3} x = x^4/24$	$D^{-3} x^2 = x^5/60$	$D^{-3} x^3 = x^6/120$	$D^{-3} x^5 = x^8/332$

Do you notice the above results? Do they look familiar? They should, they are the indefinite integral as listed below:

$$D^{-1}x^p = \int x^p dx$$
$$D^{-2}x^p = \iint x^p dxdx$$
$$D^{-3}x^p = \iiint x^p dxdxdx$$
$$\vdots \qquad \vdots \qquad\qquad\qquad \vdots$$
$$D^{-n}x^p = \int (n\text{ - times})\ x^p dx(n\text{ - times})$$

Again, I will boldly go where no graduate student have gone before and say that ***(a_6)*** is a valid for all n positive or negative as long as they are integers. It is not unreasonable to say that if $n \geq 0$ and n is an integer, then ***(a_6)*** will become the derivative formula for x^p, and if $n < 0$ and n is an integer, then ***(a_6)*** will become the indefinite integral formula for x^p.

Now, with a little help and guidance from Dr. Osler I will again make use of the gamma function, recall that z! = Γ(z + 1). Substituting p! and (p – n)! from ***(a_6)*** with the gamma function will yield:

$$D^n x^p = \frac{\Gamma(p+1)x^{p-n}}{\Gamma(p-n+1)} \qquad (a_7)$$

Since $\Gamma(\alpha)$ is defined for all values of z, integer or non-integer, I will now replace n in ***(a_7)*** with α. Rewriting ***(a_7)*** we obtain:

$$D^\alpha x^p = \frac{\Gamma(p+1)x^{p-\alpha}}{\Gamma(p-\alpha+1)} \qquad (a_8)$$

Derivatives of *f*(x)

Fueled with a general derivative for functions in the form of x^p I will derive a large number general derivative of functions in the form *f*(x). Knowing that we can expand *f*(x) in a Taylor's series and making use of ***(a_8)*** I will derive a large variety of fractional derivatives for the form *f*(x). If

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

then

$$D^{\alpha} f(x) = \sum_{n=0}^{\infty} a_n D^{\alpha} x^n$$

by "pushing" the D operator through sigma, yielding:

$$D^{\alpha} f(x) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)x^{n-\alpha}}{\Gamma(n-\alpha+1)} \qquad (a_9)$$

(a_9) is a very good candidate for functions that can be expanded in a Taylor's series.

The Big Fix

The reader has just sat through seven boring pages of simple calculus probably wondering where I am going. I first derived a possible candidate for the definition of the fractional derivatives in the form e^x which lead to an unexpected contradiction. Then I derived another possible candidate for the form x^p and another candidate for the general function $f(x)$. Well! Continue reading and I will try to clear up the contradictions.

Recalling from elementary calculus:

$$D^{-1}f(x) = \int f(x)dx$$
$$D^{-2}f(x) = \iint f(x_1)f(x_2)dx_1dx_2$$
$$D^{-3}f(x) = \iiint f(x_1)f(x_2)f(x_3)dx_1dx_2dx_3$$

and so on.

However, we obtain a problem since the indefinite integrals have arbitrary constants. I will get around this problem by using the following limits.

$$D^{-1}f(x) = \int_0^x f(x)dx$$

The double integral will be integrated from 0 to x and then from t_1 to x, integrating the double integral in this fashion can be found in many elementary calculus books.

$$D^{-2}f(x) = \int_0^x \int_{t_1}^x f(t_1)dt_2dt_1$$

We can "pull" the function $f(t_1)$ outside the second integral since it is not a function of t_2, which will yield:

$$D^{-2}f(x) = \int_0^x f(t_1)\int_{t_1}^x dt_2dt_1$$

Notice that:

$$\int_{t_1}^{x} dt_2 = x - t_1$$

Substituting this in the previous equation, and dropping the subscripts, will yield:

$$D^{-2} f(x) = \int_0^x f(t)(x-t)dt$$

Following the same process will produce the following integrals:

$$D^{-3} f(x) = \frac{1}{2}\int_0^x f(t)(x-t)^2 dt$$

$$D^{-4} f(x) = \frac{1}{2*3}\int_0^x f(t)(x-t)^3 dt$$

$$D^{-5} f(x) = \frac{1}{2*3*4}\int_0^x f(t)(x-t)^4 dt$$

In general, the nth integral will become:

$$D^{-n} f(x) = \frac{1}{(n-1)!}\int_0^x f(t)(x-t)^{n-1} dt$$

Recalling the fact that ***Γ(n) = (n – 1)!*** and as previously done I will replace – ***n*** with α which will yield.

$$D^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)}\int_0^x \frac{f(t)dt}{(x-t)^{\alpha+1}} \qquad (a_{10})$$

Notice that the above expression is looking a lot like the Cauchy's integral formula? Well, the expression is rapidly becoming the definition of a general formula for a fractional derivative, but we still need to take care of a few problems. If $\alpha < -1$ the expression is a fractional derivative and can be used as a definition.

But, if α is greater than -1 the expression becomes undefined. The reader can easily see as t approaches x, $x - t$ approaches 0 and the denominator becomes undefined. Although I have called the expression (a_{10}) a fractional derivative it is truly an integral if $\alpha \leq -1$. Therefore, (a_{10}) will involve limits, which is very uncommon when one thinks of derivatives. This is reason why we had a problem when we tried to match (a_1) and (a_5); we did not take into consideration the idea of limits. It is common in the field of fractional derivatives to us the following notation, which sets the limits of integration from b to x:

$$ {}_bD_x^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_b^x \frac{f(t)dt}{(x-t)^{\alpha+1}} \qquad (a_{11}) $$

Lets go back to the beginning of this chapter and look at (a_1) where I tried to derive a fractional derivative of the form e^{ax}, which would be valid for any n.

$$ D^n e^{ax} = a^n e^{ax} \qquad (a_1) $$

Using my new formula (a_{11}) I will try to find the limits of integration to validate (a_1) so that it will become valid for any n. We know from calculus that $D^{-1}e^{ax} = a^{-1}e^{ax}$, then:

$$ {}_bD_x^{-1} e^{ax} = \int_b^x e^{ax} dx = \frac{1}{a} e^{ax} \Big|_b^x = \frac{1}{a} e^{ax} - \frac{1}{a} e^{ab} $$

One can see that in order for (a_1) to be valid then we need to make:

$$ \frac{1}{a} e^{ab} = 0 $$

This will occur four different fashions, the first two are the trivial cases when $a = \pm\infty$. The other two cases will occur when the function e^{ab} goes to zero, this happens when $ab = -\infty$. When a is positive and b is negative, or when b is positive and a is negative.

As a side note, in equation a_{11} if a is positive, b is $-\infty$, and = $\alpha - 1$ we get what is known as the Weyl fractional derivative, or the Weyl Transform which is denoted as:

$$_{-\infty}W_x^{-1}e^{ax} = \int_{-\infty}^{x} e^{ax}dx$$

Weyl discovered this form of a fractional derivative in 1917 and his form is now widely used. Surprisingly, the Weyl Fractional Derivative can be used to solve the fractional derivative of the form e^{ax} for any α positive or negative. It is not difficult for the reader to convince themselves that the Weyl Fractional Derivative can be used to solve the fractional derivative of the form e^{ax} for any α positive or negative, therefore I will no waste the reader's time. Although, I did not use the formula (a_{11}) it did lead me to the Weyl Fractional Derivative formula by way of the idea of limits.

Finally, to the reader's delight as well as myself, I have shown that the formula a_{11} is the Weyl Fractional Derivative formula if we let $b = -\infty$,

$$_{-\infty}W_x^{\alpha}e^{ax} = a^{\alpha}e^{ax}$$

and it is valid for any α positive or negative. It is very surprising, at lease to me, that the solution involved limits, one would not expect limits to be involved with derivatives since we think of them as local properties of functions. Nevertheless, the contradiction between (a_1) and $(a5)$ is now solved.

Since limits were involved in the fractional derivative of the form e^{ax} let us look at the fractional derivative of the form x^p.

$$D^n x^p = \frac{p!}{(p-n)!}x^{p-n}$$

We know from calculus that:

$$D^{-1}x^p = \frac{x^{p+1}}{p+1}$$

and making use of *(a_{11})* will yield:

$$D^{-1}x^p = {}_bD_x^{-1}x^p = \int_b^x x^p dx = \frac{x^{p+1}}{p+1} - \frac{b^{p+1}}{p+1}$$

But we want:

$$D^{-1}x^p = \frac{x^{p+1}}{p+1}$$

Therefore, I need

$$\frac{b^{p+1}}{p+1} = 0$$

this will only happen when $b = 0$, which will yield:

$$D^{-1}x^p = {}_bD_x^{-1}x^p = \int_0^x x^p dx = \frac{x^{p+1}}{p+1}$$

Again, it is not difficult for the reader to see that formula *(a_{11})* will be valid for fractional derivatives of the form x^p for any α, positive or negative if we set the lower limit $b = 0$.

$$ {}_0D_x^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{\alpha+1}} \qquad (a_{12})$$

It should now be clear to the reader that the fractional derivative *(a_{11})* might be able to be made valid for any function that can be expanded in a Taylor's series. The "trick" to using formula *(a_{11})* is to find the lower limit b that works for the particular form of function that you are working on.

References

[2.1] Miller, Kenneth, S, and Ross, Bertram, 1993. An Introduction to the Fractional Calculus and Fractional Differential Equations, *John Wiley & Sons, Inc.* pp. 33 to 35.

[2.1] Osler, Thomas, J, Lectures from Complex Analysis Course at Rowan University, Spring Semester, 2000, Not Published.

[2.2] Osler, Thomas, J & Kleinz, Marcia, *A Child's Garden of Fractional Derivatives*, To be Published.

Chapter 3

A Rigorous Approach

Using Complex Analysis

I will begin by looking at the Riemann Liouville Integral:

$$_0D_z^{\alpha} f(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z \frac{f(t)dt}{(z-t)^{\alpha+1}}$$

This integral is wildly accepted throughout the mathematical community, provided the stipulation is given that the real part of α is less then zero, Re $(\alpha) < 0$. If Re $(\alpha)>0$ the integral will generally diverge and become useless for the needed application. α needs to be truly arbitrary with no restrictions. A nice way out of this problem is to start with the Cauchy's Integral Formula.

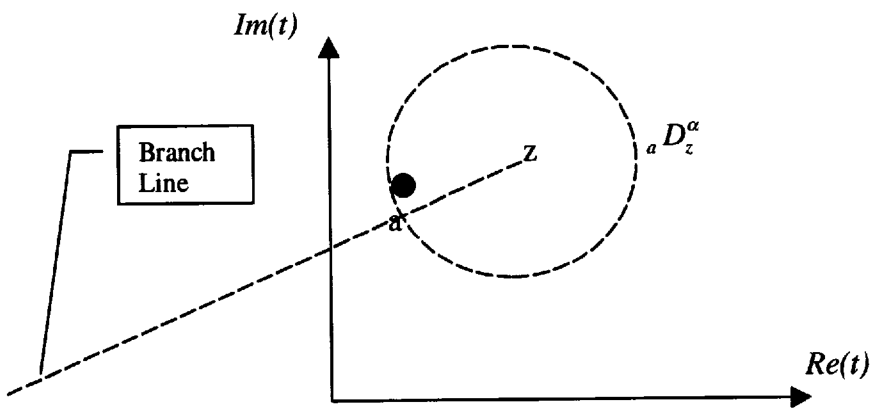
$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint \frac{f(t)dt}{(t-z)^{n+1}}$$

Replacing n with any arbitrary number α, and replacing $\alpha!$ with the gamma function will yield:

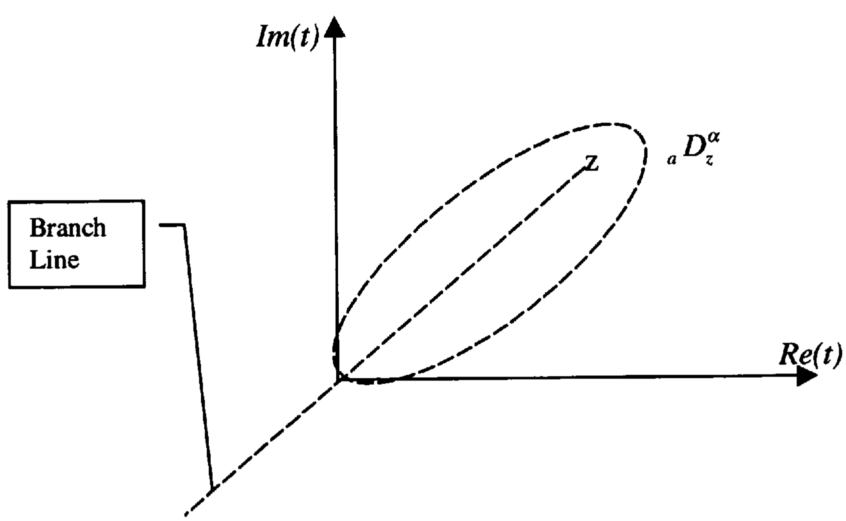
$$D^{\alpha} f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \oint \frac{f(t)dt}{(t-z)^{\alpha+1}}$$

Right away one can see that there is a problem emerging, i.e., switching to the Cauchy's Integral Formula has changed the singularities. This was precisely the problem in Chapter 2, of this paper, where ***(a_2)*** did not equal ***(a_5)***. (The reader should recall that the intuitive approach to e^x did not equal the formula approach.)

If we let α be a fractional number, (that does not simplify to a integer), then a branch line will emerge since t goes to z and there is no way to solve this kind of problem. This is the reason why equations ***(a_2)*** and ***(a_5)*** from Chapter 2, of this paper, did not match. For example, if $\alpha = \frac{1}{2}$, then the value of the integral will become dependent on where the contour crosses the branch line. I will show how to make the proper choice of the placement of the branch line which will allow the Cauchy Integral fractional derivative to generalize the Riemann Liouville Integral.



If we "swing" the branch line in such a way as to let it pass through the origin, and let a be at the origin we will be able to solve the problem.

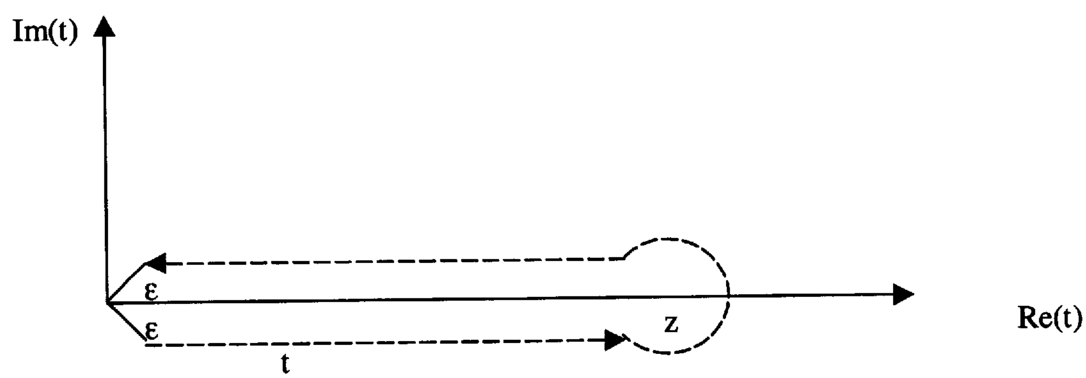


This adjustment to the branch line is denoted as $\oint_0^{z^+}$ and is defined as the closed integral starting at $t = 0$, circles $t = z$ once in a positive sense and returns to zero. The new formula will be given as:

$$ {}_0D_z^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i}\oint_0^{z^+}\frac{f(t)dt}{(t-z)^{\alpha+1}} \qquad \boldsymbol{(b_1)} $$

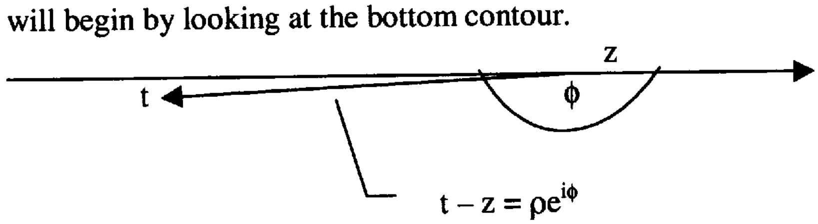
(b₁) has the advantage of not having as many restrictions on α like the Riemann Liouville Integral does, this is because t does not go anywhere near z.

The question that remains to be answer is; will $\boldsymbol{(b_1)}$ match up with the Riemann Liouville Integral? For simplicity we will begin by letting z lie on the real-axis of $\boldsymbol{t}$.



It should not be difficult for the reader to see that as ε goes to zero the contribution of the "slanted segments" to the integral goes to zero. In fact as the length of ε goes to zero the contour will lie on the real axis of $\boldsymbol{t}$. So we will do the computations as if the contour is on the real axis.

I will begin by looking at the bottom contour.



$\rho = z - t$

It is not difficult to see that $\phi = -\pi$. Therefore the bottom integral will be given as follows:

$$\text{Let: } \oint_{bottom} = \int_{\text{as stated above}} f(x)(t-z)^{-\alpha-1}dt \qquad \boldsymbol{(b_2)}$$

I must first deal with the real and imaginary parts of $(t-z)^{-(\alpha+1)}$. Using some simple rules and identities, from Complex Analysis, will yield the following.

$$(t-z)^{-(\alpha+1)} = \left(e^{\ln(t-z)}\right)^{-(\alpha+1)}$$

$$= \left[e^{\ln|t-z|+i\cdot\arg(t-z)}\right]^{-(\alpha+1)}$$

$$= \left[e^{-(\alpha+1)\ln|t-z|}\right] * \left[e^{i\cdot\arg(t-z)(-(\alpha+1))}\right]$$

Since the bottom contour is going from ***0*** to *z*, (from left to right), I will use an absolute value identity.

$$|t-z| = z-t$$

Now, continuing with the above simplification:

$$(t-z)^{-(\alpha+1)} = \left[e^{-(\alpha+1)\ln(z-t)}\right] * \left[e^{i\cdot\arg(t-z)(-(\alpha+1))}\right]$$

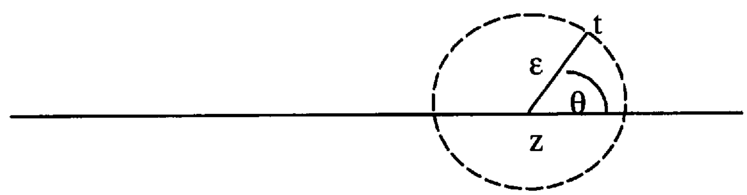
Combining the above with $\boldsymbol{(b_2)}$ and a little simplification I obtain:

$$\oint_{bottom} = e^{i\pi(\alpha+1)}\int_0^z f(t)(z-t)^{-(\alpha+1)}dt$$

Simplifying a little more, and making use of the fact that $e^{i\pi(\alpha+1)} = -e^{i\pi\alpha}$ will yield:

$$\boxed{\oint_{bottom} = -e^{i\pi\alpha}\int_0^z \frac{f(t)}{(z-t)^{(\alpha+1)}}dt}$$

It should not be difficult for the reader to see that the integral around the circle will equal zero since we will shrink the circle to nothing. Since this is a rigorous proof I will give some details of this contour.



I will begin with contour around the circle, which will be given as:

$$\oint_{circle} = \int_{\vartheta_0}^{\vartheta_0+2\pi} f(t)(t-z)^{-(\alpha+1)}$$

Making use of a few trigonometric identities will yield:

$$t = z + \varepsilon \cdot e^{i\vartheta}$$

then:

$$dt = \varepsilon \cdot i \cdot e^{i\vartheta} d\vartheta$$

Combining the last three formulas will yield:

$$\oint_{circle} = \int_{\vartheta_0}^{\vartheta_0+2\pi} f\left(z + \varepsilon \cdot e^{i\vartheta}\right) \varepsilon^{-(\alpha+1)} e^{-i(\alpha+1)\vartheta} \varepsilon \cdot i e^{i\vartheta} d\vartheta$$

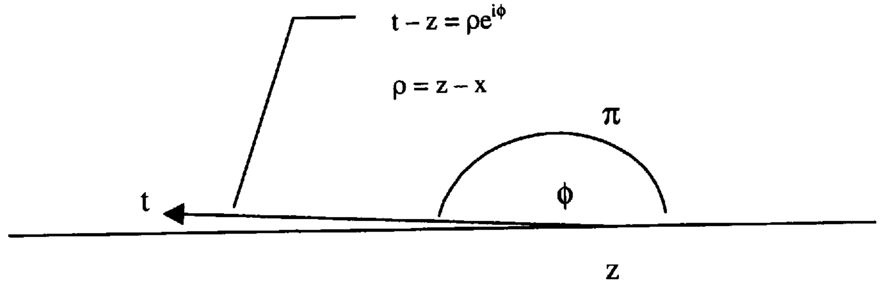
Simplifying and taking the absolute value we get:

$$\oint_{circle} \left| \int_{\vartheta_0}^{\vartheta_0+2\pi} f\left(z + \varepsilon \cdot e^{i\vartheta}\right) \varepsilon^{-\alpha} e^{-i\alpha\vartheta} i d\vartheta \right|$$

$$\leq B \int_{\vartheta_0}^{\vartheta_0+2\pi} \left| f\left(z + \varepsilon \cdot e^{i\vartheta}\right) \right| \varepsilon^{-\alpha} d\vartheta \to 0, \text{ as } \varepsilon \to 0 \text{ since } \operatorname{Re}(\alpha) < 0$$

Where $B = \max\left|e^{-i\alpha\vartheta}\right|, \ 0 \leq \vartheta \leq 2\pi$

Now we need to deal with the contour along the top.



It is not difficult to see that φ approaches π as ε goes to zero. Therefore the top integral will be given as follows:

$$\oint_{top} = \int_z^0 f(t)(z-t)^{-\alpha-1} e^{i\pi(-\alpha-1)} dt$$

Simplifying in the same fashion as I did with the bottom integral will yield:

$$\oint_{top} = e^{-i\pi(\alpha+1)} \int_z^0 f(t)(z-t)^{-(\alpha+1)} dt$$

Finally, reversing the limits of integration and using $e^{-\pi I} = 1$:

$$\boxed{\oint_{top} = e^{-i\pi\alpha} \int_0^z \frac{f(t)}{(z-t)^{(\alpha+1)}} dt}$$

Putting all of the parts together will yield:

$$\int_0^{z+} = \int_{bottom} + \int_{circle} + \int_{top}$$

$$\oint_0^{z^+} = -e^{i\pi\alpha} \int_0^z \frac{f(t)}{(z-t)^{(\alpha+1)}} dt + 0 + e^{-i\pi\alpha} \int_0^z \frac{f(t)}{(z-t)^{(\alpha+1)}} dt$$

Simplifying:

$$\oint_0^{z^+} = \left(-e^{i\pi\alpha} + e^{-i\pi\alpha}\right)\left(\int_0^z \frac{f(t)}{(z-t)^{\alpha+1}} dt\right)$$

Factoring out a *(− 1)* and multiplying the numerator and denominator by ***2i*** will yield:

$$\oint_0^{z^+} = \left(\frac{-\left(e^{i\pi\alpha} - e^{-i\pi\alpha}\right)}{2i} \right) (2i) \left(\int_0^z \frac{f(t)}{(z-t)^{\alpha+1}} dt \right)$$

Applying the trigonometric identity:

$$\sin\vartheta = \frac{e^{i\vartheta} - e^{-i\vartheta}}{2i}$$

will yield:

$$\oint_0^{z^+} = (-2i \cdot \sin(\pi\alpha)) \left(\int_0^z \frac{f(t)}{(z-t)^{\alpha+1}} dt \right)$$

The above equation is the integral part of the Cauchy's Integral formula. Inserting it into the formula will yield:

$$D^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} (-2i \cdot \sin(\pi\alpha)) \left(\int_0^z \frac{f(t)}{(z-t)^{\alpha+1}} dt \right)$$

Simplifying:

$$D^\alpha f(z) = \frac{\Gamma(\alpha+1)}{\pi} (-i \cdot \sin(\pi\alpha)) \left(\int_0^z \frac{f(t)}{(z-t)^{\alpha+1}} dt \right)$$

Fortunately, an identity of the Gamma function is:

$$\frac{1}{\Gamma(-\alpha)} = \frac{\Gamma(\alpha+1)}{\pi} (-i \cdot \sin(\pi\alpha))$$

Replacing the Gamma identity into the above Cauchy's Integral formula will yield:

$$\boxed{D^\alpha f(z) = \frac{1}{\Gamma(-\alpha)} \left(\int_0^z \frac{f(t)}{(z-t)^{\alpha+1}} dt \right)} \qquad (b_3)$$

As the reader can see in the case when $\mathrm{Re}(\alpha) < 0$ the above Cauchy's Integral formula, ***(b_3)***, is truly the Riemann Liouville Integral, but is defined with less restrictions on α. Equation ***(b_3)*** is good for all α such that $\alpha \neq -1, -2, -3, \ldots$, since the Gamma Function is not defined at these values.

If $\alpha = -1, -2, -3, \ldots$ one would then use the Riemann Liouville Integral.

References

[3.1] K. S. Miller & Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations,* John Wiley & Sons, New York, 1993.

[3.2] Osler, Thomas, J, Lectures from Complex Analysis Course at Rowan University, Spring Semester, 2000.

[3.3] Osler, Thomas, J & Kleinz, Marcia, *A Child's Garden of Fractional Derivatives,* To be published.

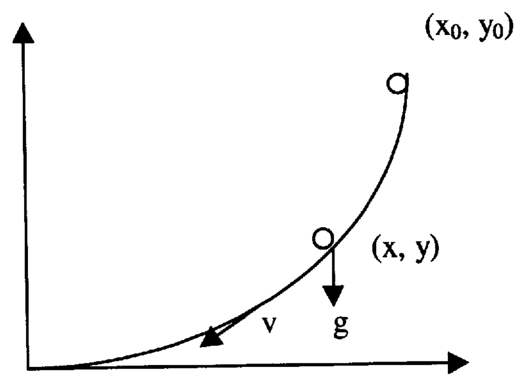
Chapter 4

The Tautochrone Problem

The tautochrone problem has been around for several hundred years and was first solved by Christen Huygens in the early 1700. However in 1823 Niels Henrik Abel [4.1] solved the tautochrone problem in an entirely different way by using fractional derivatives. In 1998 Dr. Thomas J. Osler and Dr. Eduardo Flores, [4.2] both from Rowan University, Glassboro, New Jersey, solved the tautochrone problem using fractional derivatives as well, but they solved the problem for arbitrary potentials.

Building on Abel's, Osler's, and Flores's work, the following is an excellent example of how the techniques of fractional derivatives can be used to solve a physical problem. Simply put, the tautochrone problem finds the cycloid needed to produce the curve so that a frictionless mass will slide down the curve, acting only on the force of gravity, to the origin in the same amount of time regardless of the starting point. The rules are surprisingly simple:

1) The bead slides without friction.
2) Initial velocity is equal to zero.
3) Time to reach origin is T
4) T is independent of Y_0, the initial height.



Solving the tautochrone problem is no easy task, and can not be done quickly. However, the end result is well worth the effort.

We begin with,

$$F = -\nabla V(x, y, z)$$

$$F = -(\partial V/\partial x i + \partial V/\partial y j + \partial V/\partial z k),$$

Where $V(x, y, z)$ is the potential function. Since the force of gravity acts only in the down direction the Force Field equation will become:

$$F = (\cancelto{0}{\partial V/\partial x i} + \partial V/\partial y j + \cancelto{0}{\partial V/\partial z k}) = \partial V/\partial y = g$$

Integrating both sides of the equation yields:

$$V = gy + c$$

Letting $c = 0$ we get the potential function:

$$V(y) = gy$$

Recalling the Conservation of Energy Law,

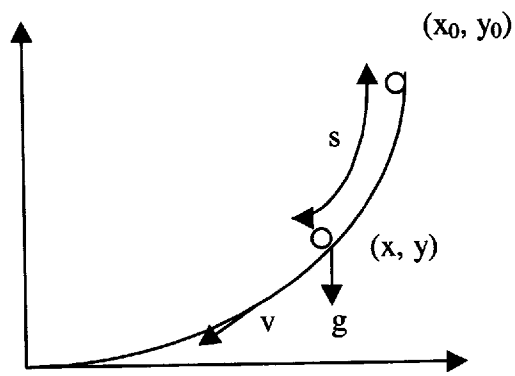
$$PE = mgy, \text{ and } KE = \tfrac{1}{2} mv^2$$

And stating that m equal the unit mass, then.

$$PE = gy, \text{ and } KE = \tfrac{1}{2} v^2$$

Fundamentally, the Conservation of Energy states that energy is conserved neither lost nor produced. Therefore, Potential Energy *(PE)* plus Kinetic Energy *(KE)* is conserved.

$$\tfrac{1}{2} v^2 + gy = \tfrac{1}{2} v_0 + gy_0$$



Since one of the initial conditions was that the initial speed is equal to zero,

$$\tfrac{1}{2} v^2 + gy = gy_0$$

$$v^2 = 2g(y_0 - y)$$

$$v = \sqrt{2g}\sqrt{y_0 - y}$$

Since the units for speed is distance/time which is a derivative of distance, and the mass is moving down the curve, the resulting equation is,

$$v = -\frac{ds}{dt} = \sqrt{2g}\sqrt{y_0 - y}$$

where s represents the arc length along the curve. Then:

$$-\frac{ds}{\sqrt{y_0 - y}} = \sqrt{2g}dt$$

Integrating the previous equation will result in:

$$\int_{y=y_0}^{y=0} \frac{-ds}{\sqrt{y_0 - y}} = \sqrt{2g} \int_{t=0}^{t=T} dt$$

Reversing the order of the integral will force a positive function, as indicated below:

$$\int_{0}^{y_0} \frac{ds}{\sqrt{y_0 - y}} = \sqrt{2g} \int_{t=0}^{t=T} dt$$

Integrating the right hand side yields:

$$\int_{0}^{y_0} \frac{ds}{\sqrt{y_0 - y}} = \sqrt{2g}T \qquad \textbf{\textit{(d}}_{\textbf{\textit{1}}}\textbf{\textit{)}}$$

Restating the general formula of fractional derivatives that we derived in Chapter 2 of this paper:

$$_0D_x^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{\alpha+1}} \qquad \textbf{\textit{(d}}_{\textbf{\textit{2}}}\textbf{\textit{)}}$$

The fractional derivative begins to emerge from the left hand side of ***(d₁)***. Using algebraic manipulation obtains:

$$\int_0^{y_0} \frac{\frac{ds}{dy}dy}{(y_0 - y)^{-1/2+1}} = \frac{\Gamma(1/2)}{\Gamma(1/2)} \int_0^{y_0} \frac{\frac{ds}{dy}dy}{(y_0 - y)^{-1/2+1}} = \Gamma(1/2) \quad {}_0D_{y_o}^{-1/2} \frac{ds}{dy} \qquad \textbf{\textit{(d}}_{\textbf{\textit{2}}}\textbf{\textit{')}}$$

Using the fact that:

$$\Gamma(1/2) = \sqrt{\pi}$$

And making use of ***(d₁)*** and ***(d₂')*** the result is:

$$\sqrt{\pi}\, {}_0D_{y_0}^{-1/2} \frac{ds}{dy} = \sqrt{2g}T$$

Dividing both sides of the equation by the square root of π will result in.

$$ {}_0D_{y_0}^{-1/2}\frac{ds}{dy}=\sqrt{\frac{2g}{\pi}}T $$

Then:

$$ \frac{ds}{dy}={}_0D_{y_0}^{1} $$

Therefore:

$$ {}_0D_{y_0}^{-1/2}\,{}_0D_{y_0}^{1}=\sqrt{\frac{2g}{\pi}}T $$

Subsequently, making use of the Law of Indexes,

$$ {}_0D_{y_0}^{1/2}=\sqrt{\frac{2g}{\pi}}T $$

And operating on both sides of the equation with ${}_0D_{y_0}^{-1/2}$, the result is:

$$ {}_0D_{y_0}^{-1/2}\,{}_0D_{y_0}^{1/2}s=\sqrt{\frac{2g}{\pi}}T\,{}_0D_{y_0}^{-1/2}1 $$

which yields:

$$ s(y)=\sqrt{\frac{2g}{\pi}}T\,{}_0D_{y}^{-1/2}1 $$

Since the object of this exercise was to find a cycloid needed to produce the curve so that a frictionless mass will slide down the curve, acting only on the force of gravity. The frictionless mass will slide to the origin in the same amount of time regardless of the starting point.

Recalling formula *(a_8)* from Chapter 2 where I defined:

$$ {}_0D_y^{1/2}1 = \frac{y^{1/2}}{\Gamma(3/2)} $$

Therefore we get:

$$ s = \sqrt{\frac{2g}{\pi}}T\frac{y^{1/2}}{\Gamma(3/2)} $$

Making use of the fact that $\Gamma(3/2) = \frac{2}{\sqrt{\pi}}$, then:

$$ s = \sqrt{\frac{2y}{\pi}}T\frac{2\sqrt{y_0}}{\sqrt{\pi}} $$

Rearranging the above equation and replacing y_0 with y will yield an equation for the distance s:

$$ \boxed{s(y) = \frac{2T\sqrt{2gy}}{\pi}} $$

I am now going to obtain x, y in terms of an angle θ.

Since:

$$ \frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2} $$

Then:

$$ \sqrt{1+\left(\frac{dx}{dy}\right)^2} = \frac{2\sqrt{2g}}{\pi}T\left(\frac{1}{2}y^{-1\backslash 2}\right) $$

Squaring both sides:

$$1+\left(\frac{dx}{dy}\right)^2=\frac{2gT^2}{\pi^2 y}$$

Subtracting 1 and taking the square root from both sides:

$$\left(\frac{dx}{dy}\right)=\sqrt{\frac{2gT^2}{\pi^2 y}-1}$$

After simplification, the result is:

$$\left(\frac{dx}{dy}\right)=\sqrt{\frac{2gT^2-\pi^2 y}{\pi^2 y}}$$

Then multiplying both sides by dy:

$$dx=\frac{1}{\pi}\sqrt{\frac{2gT^2-\pi^2 y}{y}}dy$$

And taking the integral of both sides:

$$x=\frac{1}{\pi}\int_0^y\sqrt{\frac{2gT^2-\pi^2 y}{y}}dy \qquad \textbf{(}\boldsymbol{d_3}\textbf{)}$$

Now, to the question of the usefulness of ***(d₃)***, the problem is to integrate ***(d₃)***. With the help of Dr. Osler, I consulted the Mathematical Handbook, by Murray R. Spiegel [4.3], the following "canned" integrals help:

$$\int\sqrt{\frac{px+q}{ax+b}}dx=\frac{\sqrt{(ax+b)(px+q)}}{a}+\frac{aq-bp}{2a}\int\frac{dx}{\sqrt{(ax+b)(px+q)}}$$

And:

$$\int\frac{dx}{\sqrt{(ax+b)(px+q)}}=\frac{2}{\sqrt{-ap}}Tan^{-1}\sqrt{\frac{-p(ax+b)}{a(px+q)}}$$

Combining the two previous equations yields:

$$\int \sqrt{\frac{pq+q}{ax+b}}dx = \frac{\sqrt{(ax+b)(px+q)}}{a} + \frac{aq-bp}{2a}\frac{2}{\sqrt{-ap}}Tan^{-1}\sqrt{\frac{-p(ax+b)}{a(px+q)}} \qquad \textbf{(d}_4\textbf{)}$$

This integral may seem complicated but it is possible to work with it. It also is helpful in the simplification of ***(d₃)***. Multiply both sides of ***(d₃)*** by π and rearrange the position of y.

$$x\pi = \int_0^y \sqrt{\frac{-\pi^2 y + 2gT^2}{y}}dy$$

Let:

$$a = 1$$
$$b = 0$$
$$q = 2gT^2$$
$$p = -\pi^2$$

Then, making good use of equation ***(d₄)***, will yield:

$$x\pi = \frac{\sqrt{(1y+0)(-\pi^2 y + 2gT^2)}}{1} + \frac{1(2gT^2)-0(-\pi^2)}{2(1)}\frac{2}{\sqrt{-1(-\pi^2)}}Tan^{-1}\sqrt{\frac{-(-\pi^2)(1y+0)}{1(-\pi^2 y + 2gT^2)}}$$

Simplifying the equation we get:

$$x\pi = \sqrt{(y)(2gT^2 - \pi^2 y)} + \frac{2gT^2}{\pi}Tan^{-1}\frac{\pi\sqrt{y}}{\sqrt{(2gT^2 - \pi^2 y)}} \qquad \textbf{(d}_5\textbf{)}$$

At this juncture I need to stop and review a little trigonometry. If:

$$\theta = Tan^{-1}\frac{\pi\sqrt{y}}{\sqrt{(2gT^2 - \pi^2 y)}} \qquad \textbf{(d}_6\textbf{)}$$

Then:

$$Sin\theta = \frac{\pi\sqrt{y}}{\sqrt{2g}T}$$

And:

$$Cos\theta = \frac{\sqrt{2gT^2 - \pi^2 y}}{\sqrt{2g}T}$$

From the above three equations we can see the following:

$$\pi\sqrt{y} = \sqrt{2g}T \ \sin\theta$$

Squaring both sides and dividing both sides by π^2 will yield:

$$y = \frac{2gT^2}{\pi^2}\sin^2\theta \qquad (d_7)$$

Since $\sin^2\theta = 1 - \cos^2\theta$, and $\sin^2\theta = \frac{1-\cos\ 2\vartheta}{2}$ the result is:

$$y = \frac{gT^2}{\pi^2}(1-\cos\ 2\vartheta) \qquad (d_8)$$

Combing ***(d₅)***, ***(d₆)***, ***(d₇)***, and ***(d₈)*** we will obtain:

$$x\pi = \sqrt{\left(\frac{2gT^2}{\pi^2}\sin^2\theta\right)\left(2gT^2 - \pi^2\frac{2gT^2}{\pi^2}\left(1-\cos^2\theta\right)\right)} + \frac{2gT^2}{\pi}\theta$$

Simplifying:

$$x\pi = \frac{2gT^2}{\pi}\sqrt{\left(\sin^2\theta\right)\left(1-\left(1-\cos^2\theta\right)\right)} + \frac{2gT^2}{\pi}\theta$$

Dividing both sides of the equation by π will yield:

$$x = \frac{2gT^2}{\pi^2}\left(\left(\sin\theta\right)\left(\cos\theta\right)+\theta\right)$$

Recalling from trigonometry, *sin 2θ = 2sin θ cos θ*, then:

$$x = \frac{gT^2}{\pi^2}\sin 2\theta + \left(\frac{gT^2}{\pi^2}\right)2\theta$$

Finally:

$$x = \frac{gT^2}{\pi^2}(\sin 2\theta + 2\theta) \qquad \boldsymbol{(d_9)}$$

and:

$$y = \frac{gT^2}{\pi^2}(1 - \cos\ 2\theta) \qquad \boldsymbol{(d_{10}}$$

Recalling again from trigonometry that a cycloid is defined as follows:

$$x = \rho(\sin\phi + \phi) \qquad \boldsymbol{(d_{11})}$$

$$y = \rho(1 - \cos\phi) \qquad \boldsymbol{(d_{12})}$$

Now, if we let:

$$\rho = \frac{gT^2}{\pi^2}$$
$$\phi = 2\theta$$

And if we substitute ρ and ϕ into ***(d_9)*** we get:

$$x = \rho(\sin\phi + \phi)$$

Lastly, if we substitute ρ and ϕ into ***(d_{10})*** we get:

$$y = \rho(1 - \cos\phi)$$

<u>Therefore, we have a cycloid and the tautochrone problem is solved!</u>

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Chapter 5

Oliver Heaviside

No paper on Fractional Derivatives would be complete without having a chapter on Oliver Heaviside. As I have mentioned in Chapter 1, Oliver Heaviside has become one of my "heroes." I look at things the same way as he did. I prefer to use an intuitive approach when looking at problems, rigorous proofs are not interesting to me as they were not to Heaviside. He believed in the scientific and experiential approach where one would continue to "play" with a particular problem until a solution was found. He believed that once a solution was found, and could be duplicated, there is no need to back it up with a rigorous proof, the solution was the solution and there is no need to go any further. He did not hate mathematics, nor do I, he just believed that rigorous proofs hindered the physicist. (Note: All references made in this Chapter will come from Heaviside's book, *Electromagnetic Theory,* republished in 1971.) Heaviside says, pp. 4, "...mathematics being fundamentally an experimental science, like any other, it is clear that the Science of Nature might be studied as a whole, the properties of space along with the properties of the matter found moving about therein. This would be very comprehensive, but I do not suppose that it would be generally practicable, though possibly the best course for a large-minded man. Nevertheless, it is greatly to the advantage of a student of physics that he should pick up his mathematics along with his physics, if he can. For then the one will fit the other. This is the natural way, pursued by the creators of analysis. If the student does not pick up so much logical mathematics of a formal kind, he will, at any rate, get on in a manner suitable for progress in his physical studies...Now, in working out physical problems there should be, in the first place, *no pretence of rigorous formalism*. The physics will guild the physicist along some-how to useful and important results, by the constant union of physical and geometrical or analytical ideas. *The practice of eliminating the physics by reducing a problem to a purely mathematical exercise should be avoided as much as possible*. The physics should be carried on right through, to give life and reality to the problem, and to obtain the great assistance, which the physics gives to the mathematics..."

Heaviside did not dislike mathematics. He questioned its usefulness to most physical problems. He believed that a rose is just a rose and no further explanation is needed. He further says, pp. 7, "...The best result of mathematics is to be able to do without it. To show the truth of a paradox by example, I would remark that nothing is more satisfactory to a physicist than to get rid of a formal demonstration of an analytical theorem and to substitute a quasi-physical one, or a geometrical one free from co-ordinate symbols, which will enable him to see the necessary truth of the theorem, and make it be practically axiomatic..."

Heaviside and myself are not alone on this subject; he reproduces a passage from the Preface to Lord Rayleigh's book on Treatise on Sound, pp. 5, which says, "...In the mathematical investigation I have usually employed such methods as present themselves naturally to a physicist. The pure mathematician will complain, and sometimes with justice, of deficient rigor. But to this question there are two sides. For, however important it may be to maintain uniformly high standard in pure mathematics, the physicist may occasionally do well to rest content with arguments, which are fairly satisfactory and conclusive from his point of view. To his mind, exercised in a different order of ideas, the more severe procedure of the pure mathematician may appear not more but less demonstrative. And further, in many cases of difficulty to insist upon the highest standard would mean the exclusion of the subject altogether in view of the space that would be required…"

Heaviside's lack of a formal mathematical schooling allowed him to do manipulations on equations in such bazaar fashions that no mathematician in his right mind would ever dream of doing. The following is just one example of how he solved a fractional derivative that, of course, is not mathematically sound. As a side note, the notation that I will use is not the same as Heaviside's since his will not fit the general notation of this paper.

Heaviside studied the voltage and current in a cable. The situation is partially based on an analogy with diffusion of heat in a rod. He used the following equations to relate V and C, *pp. 30-31*:

$$-\frac{dC}{dx}=\frac{d}{dt}V$$

and:

$$-\frac{dV}{dx}=RC$$

He combined the above two equations to give:

$$\frac{d^2V}{dx^2}=R\frac{d}{dt}V \qquad \textbf{(H)}$$

He then introduced q to "satisfy" the above equation:

$$q^2 = \frac{d}{dt}V$$

That is, Heaviside essentially let:

$$q = \left(\frac{d}{dt}\right)^{1/2},$$

so that equation ***(H)*** becomes:

$$\frac{d^2V}{dx^2} = Rq^2V$$

He proceeded to "solve" ***(H)*** by assuming q was a constant, (see pp. 4). This led Heaviside to the problem of determining the meaning of $D^{1/2}$ **1**, where **1** represents what is now referred to the "Heaviside Function."

The following account of Heaviside's solution is taken from his book, *Electromagnetic Theory, pp. 286-288.* Heaviside wanted to know the current in a wire at any point along such wire. He knew from experiments that the current at any point along a very long wire satisfied:

$$D^{1/2}C = \frac{1}{\sqrt{\pi t}}$$

Heaviside started with the following notation and two equations:

- R = resistance, per unit length of the wire.
- S = permittance per unit length.
- V(x) = voltage at any point along the wire.
- C(t) = current in the wire at time t, where for $t > 0$, C(t) is assumed to be one.
- E = impressed voltage.
- C_0 = current at x = 0.

$$C_0 = \sqrt{\frac{S}{R\pi t}}E$$
$$C_0 = \sqrt{\frac{SC(t)}{R}}E$$

Heaviside set the two equations equal to each other, which yielded:

$$\sqrt{\frac{S}{R\pi t}}E = \sqrt{\frac{SC(t)}{R}}E$$

Multiplying and dividing both sides by *E, R,* and *S* yielded:

$$\sqrt{C(t)} = \sqrt{\frac{1}{\pi t}} \qquad e_1$$

At this point in time I must point out that the above three lines of work was pieced together from a few hundred pages of Heaviside's work. He knew that the end result would be a fractional derivative in the form,

$$D^{\frac{1}{2}}C = \sqrt{\frac{1}{\pi t}}$$

since he stated, pp. 286, "...In order to avoid introducing the idea of fractional differentiation from the theoretical standpoint, I took the value of $D^{1/2}C$ as known experimentally...There is no question as to its value; that is settled by Fourier's investigation in the theory of the diffusion of heat in conductors..."

His next step, from what I could tell, was not to square both sides of the equation, but to just simply "push" the ½ power through the function e_1, which yielded:

$$D^{1/2}C = (\pi t)^{-1/2} \qquad e_2$$

The above is most definitely not mathematically logical, and it is the only step that Heaviside could have taken to get from e_1 to e_2.

Nevertheless he did give a semi-convincing formal proof using sound mathematics. He started with a fractional derivative, and used a technique which he called, pp. 288, "…an ingenious device, also well-known…"

$$\left(D^{1/2}C\right)^2 = \frac{2}{\pi}\int_0^{\infty} e^{-x^2 t}dx * \frac{2}{\pi}\int_0^{\infty} e^{-y^2 t}dy$$

$$\left(D^{1/2}C\right)^2 = \frac{4}{\pi^2}\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)t}dxdy$$

$$\left(D^{1/2}C\right)^2 = \frac{1}{\pi^2}\int_0^{\infty} 2\pi r dr e^{r^2 t}$$

$$\left(D^{1/2}C\right)^2 = \frac{1}{\pi t}\left[-e^{-r^2 t}\right]_0^{\infty}$$

$$\left(D^{1/2}C\right)^2 = \frac{1}{\pi t}$$

$$D^{1/2}C = (\pi t)^{-1/2}$$

I suppose that Heaviside was not totally convinced with this short proof, because he gave a second proof-containing trigonometry, which I will reproduce as follows.

Heaviside began with an infinitely long wire that was subjected at one end with an impressed voltage E; the current produced was expressed by:

$$C = \left(S\frac{[D^1 C]}{R}\right)^{1/2} E \qquad \boldsymbol{H_1}$$

Heaviside said, pp. 287, "…If E is constant, we have to find what $D^{1/2}C$ means. Now, we can work out this problem in Fourier series, first for a finite cable, and then proceed to the limit…"

He let the wire be of length l and be earthed at one end and have an impressed voltage E at the other. The capacity of the skin of the wire s is given by:

$$s^2 = -RS\left(D^{1/2}C\right)$$

The voltage anywhere on the wire, at any x, due to E is given by:

$$V = \frac{\sin s(l-x)}{\sin sl}E \quad ,$$

where $V = E$ at the beginning of the wire and $V = 0$ at the end of the wire. Heaviside then used one of his expansion theories, which yielded:

$$V = E\left(1-\frac{x}{l}\right)-\frac{2E}{\pi}\sum\frac{\pi}{l}\frac{\sin sx}{s}e^{-\frac{s^2t}{RS}}$$

Where s has the values of π/l, 2π, $3\pi/l$, ... As l becomes infinitely large the previous equation is converted from a Fourier series to a definite integral as follows:

$$V = E-\frac{2E}{\pi}\int_0^\infty ds\frac{\sin sx}{s}e^{-\frac{s^2t}{RS}}$$

Heaviside then says, pp. 288, "... The current at the beginning, $x = 0$, is got by

$$C = -\left(\frac{dV/dx}{R}\right)$$

and then putting $x = 0$. This makes

$$C = \frac{2E}{R\pi}\int_0^\infty e^{-\frac{s^2t}{RS}}ds \qquad \boldsymbol{H_2}$$

Heaviside's next step was to set H_1 and H_2 equal to each other, which would have yielded:

$$\left(S\frac{[D^1C]}{R}\right)^{1/2} E = \frac{2E}{R\pi}\int_0^\infty e^{-\frac{s^2t}{RS}}ds \qquad e_3,$$

he then used some substitution rules from calculus:

$$u^2 = \frac{S^2}{RS} \rightarrow u = \frac{S}{\sqrt{RS}} \rightarrow du = \frac{dS}{\sqrt{RS}}$$

He said, pp. 288, "...Comparing H_1 and H_2 and removing unnecessary constants, we see that

$$D^{1/2}C = \frac{2}{\pi}\int_0^\infty e^{-u^2t}du \qquad H_3$$

Which is a well-known integral..." At this point Heaviside uses the first proof that I reproduced to arrive at the required result:

$$D^{1/2}C = (\pi t)^{-1/2}$$

Let us take a closer look at how Heaviside arrived at H_3. He said, "...Comparing H_1 and H_2 and removing unnecessary constants..." which is e_3. I agree with removing unnecessary constants to simplify an equation, which will yield:

$$\left(D^1C\right)^{1/2} = \frac{2}{\pi}\int_0^\infty e^{-s^2t}ds$$

The only way Heaviside could arrive at H_3 was to "push" the square root through the derivative to obtain a fractional derivative. Although Heaviside never stated the step he took to arrive at H_3 one can easily see the unsound mathematical steps that he took.

Heaviside made a blanket statement to justify the above proof, he said, pp. 288, "...The above is only one way in a thousand. I do not give any formal proof that all ways properly followed must necessarily lead to the same result…" I am not sure what the reader thinks, but I have always been under the impression that for a proof to be mathematically sound it must be capable of being reproduced in another fashion so that the end results are the same, or in the same form.

In closing of this Chapter I suppose that the reader and myself could debate and discuss Heaviside's methods for years without end. But Heaviside can not be denied the great accomplishments that he made during his years of being a scientist. Although I am not fond of rigorous proof, I do know, and agree, that they are needed. What intrigued me the most was the fact that Heaviside was able to make such great accomplishments in electromagnetism as well as differential equations without any formal mathematical studies. This is why I called him one of my "heroes".

Lastly, I will add that many mathematicians have been motivated to make some of Heaviside's arguments rigorous. The reader can consult Hillel Poritsky's expository account in the MAA Monthly to learn more about this matter.

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Conclusion

In Chapter 1 of this paper the reader has seen the fact that not many mathematicians have even attempted to work on the topic of fractional derivatives. Most of the ones that did worked on the topic only did so briefly, and many acknowledged the existence but did nothing with the topic. More work has been done on fractional derivatives in the last fifty years then was done before that time. I feel that it is a fascinating subject, but has been overlooked far to long by the mathematical community.

In Chapter 2 the reader gets a small look into my mind to see how I feel about using the intuitive approach. I have been blessed with the ability to "toy" with mathematical problems and come up with valid solutions to them. Unfortunately, I can not always back them up with rigorous proofs. Although, this particular topic I have become very good at backing up my work with rigor by using Complex Analysis, as the reader was able to see in Chapter 3.

Chapter 4, The Tautochrone Problem, is another example of my newfound ability to back up my intuitive approaches with rigor by means of Complex Analysis. I have found that many proofs can be done more easily with the help of Complex Analysis than as they are commonly done. In fact undergraduates can understand many of these proofs more easily when done in the Complex Analysis form.

Lastly, in Chapter 5 the reader was able to see my views on Oliver Heaviside. Although, I do not agree with the methods he used in solving some of his problems, I do understand what he was trying to accomplish. He knew the solutions to many of his problems through experimental data, but he just did not have the mathematical training needed to back up his work with rigor. Please do not get the wrong impression, I do not agree with the unsound mathematical logic that Heaviside used, but I do agree with using the intuitive approach whenever possible.

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