The development of a mini course on fractal geometry for the community college mathematics student

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THE DEVELOPMENT OF A MINI COURSE ON
FRACTAL GEOMETRY FOR THE
COMMUNITY COLLEGE
MATHEMATICS
STUDENT

by
Kananathan Satchithananthan

A Thesis

Submitted in fulfillment of the requirements of the
Master of Arts Degree in the Graduate Division
of Rowan College in Mathematics
Education 1996

Approved by

John Sooy

Date Approved

ABSTRACT


The purpose of this study was to construct a mini introduction to a fractal geometry course appropriate for community college students. The researcher evaluated the related research and the literature related to fractal geometry appropriate for community college students. Fractal geometry lesson plans were constructed. The topics covered in this introduction course include the Koch Curve, Sierpinski’s Triangle, iterations, complex numbers and the Mandelbrot set.

The course is intended for students who have completed Calculus I at Atlantic Community College which is located in the southern part of New Jersey. The course is intended to be taught for five weeks for two hours of instruction per week. Exercises based on the material are explained and assigned to the students at the end of each class.
MINI-ABSTRACT


The purpose of this study was to construct a mini introduction to a fractal geometry course appropriate for community college students. The course is intended to be taught for five weeks for two hours of instruction per week. Exercises based on the material are explained and assigned to the students at the end of each class.
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Introduction to the Study</td>
<td>1</td>
</tr>
<tr>
<td>Introduction and Background</td>
<td>1</td>
</tr>
<tr>
<td>Problem</td>
<td>3</td>
</tr>
<tr>
<td>Significance of the Problem</td>
<td>3</td>
</tr>
<tr>
<td>Limitations</td>
<td>5</td>
</tr>
<tr>
<td>Definitions</td>
<td>6</td>
</tr>
<tr>
<td>Procedure</td>
<td>9</td>
</tr>
<tr>
<td>2. Review of Related Research and Literature</td>
<td>10</td>
</tr>
<tr>
<td>Introduction</td>
<td>10</td>
</tr>
<tr>
<td>Review of Related Research</td>
<td>10</td>
</tr>
<tr>
<td>Review of Related Literature</td>
<td>18</td>
</tr>
<tr>
<td>3. Procedure</td>
<td>24</td>
</tr>
<tr>
<td>Introduction</td>
<td>24</td>
</tr>
<tr>
<td>Selection of Related Materials</td>
<td>24</td>
</tr>
<tr>
<td>Selection of Related Fractal Topics</td>
<td>25</td>
</tr>
<tr>
<td>Construction of Individual Lesson Plans</td>
<td>26</td>
</tr>
<tr>
<td>4. Five Week Mini Course in Fractals</td>
<td>29</td>
</tr>
<tr>
<td>Introduction</td>
<td>29</td>
</tr>
<tr>
<td>Chapter</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>------</td>
</tr>
<tr>
<td>Description of Lesson Plans</td>
<td>29</td>
</tr>
<tr>
<td>Lesson Plans</td>
<td>30</td>
</tr>
<tr>
<td>Lesson Plan 1: Fractal Pictures</td>
<td>31</td>
</tr>
<tr>
<td>Lesson Plan 2: Iterations A Geometrical Approach</td>
<td>41</td>
</tr>
<tr>
<td>Lesson Plan 3: Functions</td>
<td>48</td>
</tr>
<tr>
<td>Lesson Plan 4: Iterations Using Scientific Calculator</td>
<td>52</td>
</tr>
<tr>
<td>Lesson Plan 5: Orbits and Fixed Points</td>
<td>56</td>
</tr>
<tr>
<td>Lesson Plan 6: Fractal Dimensions</td>
<td>64</td>
</tr>
<tr>
<td>Lesson Plan 7: Introduction to Complex Numbers</td>
<td>70</td>
</tr>
<tr>
<td>Lesson Plan 8: Plotting Complex Numbers</td>
<td>75</td>
</tr>
<tr>
<td>Lesson Plan 9: Mandelbrot Set</td>
<td>79</td>
</tr>
<tr>
<td>Lesson Plan 10: Mandelbrot Set Computer Experiments</td>
<td>85</td>
</tr>
<tr>
<td>5. Summary of Findings, Conclusion, and Recommendations</td>
<td>89</td>
</tr>
<tr>
<td>Introduction</td>
<td>89</td>
</tr>
<tr>
<td>Summary of Findings</td>
<td>89</td>
</tr>
<tr>
<td>Conclusion</td>
<td>90</td>
</tr>
<tr>
<td>Recommendations</td>
<td>90</td>
</tr>
<tr>
<td>Bibliography</td>
<td>91</td>
</tr>
</tbody>
</table>
CHAPTER 1

Introduction to the Study

Introduction and Background

Fractal geometry and chaos theory are revolutionizing mathematics and science, and providing us with new ways of seeing reality. A fractal is a geometrical figure which repeats itself on an ever diminishing scale. Once fractals were just abstract mathematical objects, but now they are receiving a lot of attention from various fields of study. Mathematical topics don't attract anyone but pure mathematicians unless they have an application. Currently the subject of fractals is in the phase of transitioning from abstract to application.

For teaching mathematics, the tools that have been used in the past are mostly paper and pen. This is no longer true for teaching fractals. To see fractals, we are depending heavily on micro computers and on taking the micro computers to their limits. Sophisticated computers will provide us with a better quality image of the fractal. To learn about fractals and chaos, computers are the appropriate tools.
Currently the subject of fractals is slowly being introduced to undergraduates, but most community college students are not exposed to the subject of fractals. One of the objectives at the community college is to prepare the students for further education. Fractals is one of the areas in mathematics in which students will need to further their education. By presenting current and new technology in the community college curriculum, it may serve to motivate and excite students about mathematics.

This study is being done in order to create a mini introduction course in fractals for community college-level students, so that in the future students will appreciate the concept of a fractal. The topics will be collected and arranged so that students who have a mathematical knowledge of Calculus-I will be able to understand these concepts. This course will be created using the information from currently available books, periodicals, literature, articles on the internet, and interviews which are related to fractals.
**Problem**

The purpose of the study is to create a five week mini course in fractals for community college mathematics students.

**Significance of the Problem**

Fractals is a new field in mathematics and computer technology has brought it to new life in the past decade. The subject of fractals is attractive to many in the field of science. “From geologists to astronomers, from weather forecasters to financial forecasters, from fine artists to pure mathematicians: virtually no field is left untouched by this radical shift in how we think, work, and create.” (Oliver 1992). Fractals have the potential to help us find a better solution for some of our present problems. “A few things that fractals model better are: plants, weather, fluid flow, geologic activity, planetary orbits, human body rhythms, animal group behavior, socioeconomic patterns...and the list goes on.” (Oliver, 1992) The subject of fractals is a growing field in the mathematical community but it is not taught at the community college level yet. Few, if any community colleges offer a course in fractals for their students.
“Plato postulated a world of ideal forms, where these perfect shapes resided unblemished.” (Wegner & Peterson, 1991)

Classical mathematics defined objects made of straight lines, squares, rectangles and circles, but in the real world these shapes don’t fit in most cases. Clouds, mountains, rivers, waterfalls, weather and sunflower fields can be described with great accuracy using fractals. When modeling nature, the fractal gives a better model than classical geometry.

Fractals often provide a more compact method of recording complex images and data than linear vectors. A fractal curve can be found that will fit any set of data. Fractals can easily represent similar forces acting on many levels of scale, where as linear geometry cannot. Fractals can be used to construct useful models of inherently unpredictable and chaotic systems, where as linear equations fail entirely.

Since the subject has a lot of potential for use in practical applications, it is important that our students have an introduction to the subject of fractals.
Limitations

There is a limited amount of information on fractals at this time, especially at the beginning level. The following sources of information are being used to create this mini-course on fractals:

1. Books about fractals and books about fractals that are being used as textbooks in four year colleges and universities.
2. Articles on fractals from educational and scientific journals that are appropriate.
3. Useful information about fractals and chaos on the internet, which is in the form of text, programs and hypertext.

This mini-course is intended to be taught in five weeks and for three hours of instruction per week for a total of ten classes. The audience should have completed Calculus-I or achieved an equivalent mathematics background. The important mathematics concepts will be reviewed quickly with examples, but not taught as in a regular mathematics course. The audience should be accustomed to using Microsoft Windows applications.

Computers will be necessary to construct fractals because fractals are nonlinear and they cannot be constructed by smooth pen strokes.
Definitions

Affine Self-similar: A set $S$ is called affine self-similar if $S$ can be subdivided into $k$ congruent subsets, each of which may be magnified by a constant factor $M$ to yield the whole set $S$.

Chaos: Chaos is apparently unpredictable behavior arising in a deterministic system because of great sensitivity to initial conditions. Chaos arises in a dynamical system if two arbitrarily close starting points diverge exponentially, so that their future behavior is eventually unpredictable.

Escape Time Algorithm: A technique for displaying maps of iterated functions; points are colored according to how quickly they escape a region when the function is iterated. (Oliver, 1992)

Fixed Point: A fixed point is a point $x_0$ that satisfies $F(x_0) = x_0$.

FRACrINT 1.18: This is shareware software that creates fractals.
Fractals: A fractal is a rough or fragmented geometric shape that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole. Fractals are generally self-similar and independent of scale.

Fractal Dimension: A measure of the irregularity, or roughness, of a shape; the degree to which the shape "fills space". No one has yet found a consistent definition of fractal dimension, let alone a universal formula for computing it. Generally, however, it is estimated by taking the logarithmic ratios of some measurable property on varying scales. (Oliver, 1992)

Iteration: Repeat a process over and over.

Julia Set: (1) Any set containing only points that remain stable during the iteration of a function. (2) Specifically, a Julia set associated with the function 

\[ Z_{n+1} = Z_n^2 + C, \] 

where \( c \) is an arbitrary constant.

Mandelbrot set: (1) The set of points that do not escape to infinity when the function \( Z_{n+1} = Z_n^2 + C \) is iterated, where \( c \) is a point itself and \( Z \) starts at origin \((0,0)\).
(2) More generally, any set of points that do not escape to infinity when a function including the point itself as a constant is iterated. A Mandelbrot set is always a map of a finite number of related Julia sets.

**Newton’s Approximation Method:** Method that is used to find the roots of an equation by guessing an initial value and using an iteration process.

\[ x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \]

**Orbits:** The sequence of points \( P_0, P_1, P_2, \ldots \) is usually called the orbit of \( P_0 \) and \( P_0 \) is called the seed of the orbit.

**Orbits:** Given \( x_0 \in \mathbb{R} \), we define the orbits of \( x_0 \) under \( F \) to be the sequence of points \( x_0, x_1 = F(x_0), x_2 = F^2(x_0), \ldots, x_n = F^n(x_0), \ldots \). The point \( x_0 \) is called the seed of the orbit.
Procedure

The topics for this course will be researched using the following methods:

1. Read and analyze currently available books, periodicals and articles on the internet which are related to fractals.
2. Read and analyze books being written by authors who have taught dynamical systems for undergraduates.
3. Review the course materials that involve introducing fractals appropriate to community college students.
CHAPTER 2

Review of Related Research and Literature

Introduction

This chapter is divided into three sections. They are the introduction, related articles, and related literature. Since the subject of Fractals is a new field, currently there is not a lot of research being done at the community college level. The related articles section explains the sequence in which fractals have been introduced to students at various levels similar to the course being created in this study. The related research section explains other articles and texts that were fundamental in creating a course at the community college level.

Through the internet the researcher was able to find information on fractal geometry courses taught at universities. Also located on the internet are computer programs for generating and experimenting with fractal images. The information found here will be expanded on, in the related literature section of this paper.

Review of Related Research

There are two articles from The College Mathematics Journal that the researcher finds relevant to the study.
In a 1991 article, Willson, describes his experience in teaching a one-semester course called "The Mathematics of Fractals". The prerequisites are three semesters of calculus plus knowledge of Pascal. Textbooks utilized in this course were *Fractals Everywhere* by Michael Barnsley and *The Beauty of Fractals* by H.O. Peitgen and P.H. Richter. It is explained that the course began with a two-week lesson on metric space topology. The author then suggests introducing geometric mapping including rotation, reflection, rescaling and translation. Next the students studied the Contraction Mapping Theorem followed by iterated function systems and attractors. It is stated that the "fractals" used in the course were all attractors of iterated function systems. The students are guided through the random iteration method and then asked to reverse the process. The author then presented several different definitions of fractal dimension and proceeded to cover chaotic dynamics on fractals. The students next studied the basic properties of the Julia set and discussed methods for drawing pictures of it. The Mandelbrot set was introduced and examined. The author reports the one-semester class is very workable and students reactions are uniformly enthusiastic.

In another 1991 article, Nievergelt, presents two sets of exercises for designing algorithms for plotting filled Julia sets and Julia sets. The first set of exercises requires High School level mathematics, complex arithmetic and the concepts of bounded and unbounded planar sets. The author begins by explaining
iterations of complex quadratic polynomials. It is stated that high schoolers
would have no trouble plotting iterations on a computer screen coloring points
that trigger bounded sequences, thus coloring the filled Julia set. The algorithm
for the filled Julia set is established, the Mandelbrot set is explained and students
are asked to determine which points belong to the Mandelbrot set. The second set
of exercises is described as an application on complex analysis and has
prerequisites of three semesters of Calculus, plus knowledge of operations on
complex numbers. The author explains that these exercises investigate the
boundary of the filled Julia set, which produces the Julia set. An algorithm is
generated using the "Inverse Iteration Method". The students are then required to
write a program which generates and colors the Julia set.

There are five articles found in the Mathematics Teacher Journal relevant
to this study, which are listed chronologically.

In a 1990 article, Kern and Mauk present a formalized lesson plan utilizing
the Logo language to generate fractals. The prerequisites are high school algebra
and geometry. It is stated that the main concept in generating fractals with turtle
graphics in Logo is recursion. The following sequence is suggested by the
authors. They present self-similarity first using real life images such as a tree and
a fern. They next demonstrate the generation of the Koch snowflake using Logo.
The procedure is outlined in detail. The students are then asked to generate the
quadric Koch curve up to level two as an exercise. Consideration next turns to fractal dimension and it is defined as how much a curve turns or how jagged an edge is with a coastline as a practical illustration. It is suggested that as an exercise the students try to calculate the fractal dimension of a boundary of the state in which they live. The authors explain how important it is for the students to be exposed to real life examples of fractals. The formula for fractal dimension is given next and the use of logarithms is practiced.

In a 1990 article, Barton explains that he was interested in the success Michael Barnsley and his colleagues at the Georgia Institute of Technology have had using affine transformations to create computer images of objects that occur in nature. The author decided to introduce his high school students to some of the basic ideas of Barnsley's work. Barton reports the method of creating images is based on a process called the "chaos game". It is explained that the chaos game consists of randomly selecting affine transformations and performing them on points in a plane. Barton first had the students play a specific version of the chaos game which generates the Sierpinski gasket. The students perform the process using paper and pencil to encourage understanding of the algorithm. The author next asked the students to predict the image they thought would be generated and encouraged those who could program to write a computer program that performed the process to verify their predictions. Barton then explained to his class a
generalized version of the chaos game using matrix equations for the transformations. The author suggests encouraging students to experiment with the chaos game by trying different transformations. Attention is then turned to The Collage Theorem, which begins with an image and the students are to find the transformations that would create the image. It is reported that as students played the chaos game and experimented, they were very excited about being able to create their own fractal images.

In a 1991 article, Frantz and Lazarnick present a seven day lesson plan describing materials and methods used to introduce second-year-algebra and precalculus classes to the Mandelbrot set. The Mandelbrot set is defined and the materials used are listed, including computers, a program written in Pascal to generate the Mandelbrot set, and a movie entitled, *Nothing But Zooms*. Days one through three were spent introducing complex numbers, practicing complex number operations, graphing complex numbers, and calculating the modulus. Day four began with describing the Mandelbrot set in broad terms, explaining iteration, generating Koch's snowflake using iteration, and finally, explaining the iterative process of the Mandelbrot set experimenting with several complex points. Day five was spent working in the computer laboratory determining whether points are in or out of the Mandelbrot set. During days six and seven, the class discussed their results, played with the Pascal program, zooming in on
different points and then watched *Nothing but Zooms*. The author reports that although the calculations were difficult for the students, they believe the students gained a good understanding of the Mandelbrot iterative process. Franz and Lazarnick state that the students found this to be an exciting experience.

In a 1991 article, Camp outlines a procedure for presenting the Koch curve to a precalculus class, but suggests that the presentation could be modified for lower levels providing the teacher introduces elementary geometry concepts (perimeter, area, surface area, volume and properties of similarity) and convergence of infinite sequences and series prior to presenting the Koch curve. Camp believes the procedure is an excellent way to introduce the iterative processes used to describe other fractals, which, if encouraged, students can investigate on their own. The author has found that students are excited by the conjectures they can make about related figures. The author's suggested procedure begins by explaining the iterative process used to generate the Koch curve up to the fifth generation with the aid of a computer for image representation. The formulas for calculating side length, number of sides, perimeter and area are developed and the divergence of the perimeter and the convergence of the area to a value less than one square unit are explained. Camp states that the students naturally found this result to be both fascinating and puzzling. The author then asks the students to guess what would happen if they
extended the procedure to a three-dimensional counterpart. The three-dimensional model is developed with the goal of generating the sequence for the surface area and the series for the volume of successive iterations. It is suggested that the instructor use tables to demonstrate the generation of the formulas and a paper model of the solids generated. Camp suggests encouraging students to write their own computer programs to generate the two-dimensional Koch curve, but includes a program written in Pascal for use in the classroom.

In a 1993 article, Coes explains a tactile method of exploring fractals and their dimensions by using tiles for building two-dimensional objects and cubes or blocks for building three-dimensional objects. The author states that self-similarity is the key to understanding fractals and is a key strand in high school geometry. The author believes students can easily build fractal models with common classroom materials and gain a strong sense of self-similarity. The method begins with a tactile exploration of similarity. Coes suggests asking the students to make an object similar to, but twice as large as a two-dimensional figure made from tiles. The author believes the manipulative promotes a depth of understanding that will not occur as readily by using just pencil and paper. It is suggested the class next be asked to build an object similar to, but twice as large as a three-dimensional figure. Next, the Sierpinski gasket is generated using tiles and the patterns used to generate self-similar objects are explained. Coes next
introduces three dimensional fractals by building the Menger sponge from interlocking cubes. Fractal dimension is explained by relating it to the scale and to the number of pieces, or generators, used to build an object similar to it, but on a larger scale than the generator. The formula for fractal dimension is examined thoroughly. Coes explains one downfall of this method. The models built are not true fractals because they do not show self-similarity at every level. However, the author feels that the shortcomings of the model provide a good discussion topic for the students and that the tactile experience is irreplaceable.

In a 1992 article in The Science Teacher, Marks describes how he introduced fractal geometry to his high school physics class. Marks dedicated three 45-minute periods to this topic, having his students use two different techniques to calculate the length and fractal dimension of the Massachusetts coastline. During the first period the students employed the compass method by using different spans of the compass to measure the length of the coastline. The fractal dimension was then calculated. The second period was used to measure the length and fractal dimension of the Massachusetts coastline utilizing the grid method. The author feels the grid method is more precise and that it reinforces and applies previously learned concepts, such as logarithms and the use of log-log paper. The final period was spent analyzing and discussing the results. Marks explains that he has provided the students with a hands-on introduction to fractal
geometry and that the greatest advantage of this lab is that it introduces students to
an intimidating subject in a non-threatening way.

Vojack (1989), prepared three lessons intended to be used as an
introduction to fractals for high school students. The prerequisites for this mini-
course are concepts of functions and operations on complex numbers. The
lessons are designed to be included in a high school pre-calculus class. Lesson
one consists of operations on complex numbers, finding the magnitude of a
complex equation, finding the conjugate of a complex equation, and composition
of complex functions. Lesson two teaches the students iteration and explains
orbits and fixed points. Lesson three presents the Julia sets, discussing notions of
attractor, repeller, basin of attraction and the Julia set. Exercises and examples are
included with each lesson. Following the three lessons the author introduces the
generation of the Sierpinski triangle and the Koch curve followed by the notion of
fractal dimension.

Review of Related Literature

There is an article in the August 1990 Scientific American which the
researcher found important to this study. Though most of the article is dedicated
to John E. Hutchinson's creation of the multiple-reduction copying machine and
its use in producing fractals, the article gives a good history of fractals. The
history explains Mandelbrot’s role in fractal research and gives real life examples of chaotic events such as weather patterns, turbulence in the atmosphere, and the beating of the human heart. The author also explains the connection between fractal geometry and chaos theory. Finally, the article explains the work of Julia and Fatou by describing how they discovered the Julia set in their study of non-linear dialects.

In a 1991 article in the *Mathematics Teacher*, Bannon, explains self-similarity and connects it to nature. He says "Whenever nature proves to be self-similar, a fractal will usually furnish a good model of it". Bannon takes a geometric approach using an explanation of transformations to explain fractals. This is not the approach taken in this study, however, relevant to this study are Bannon’s explanations of self-similarity, the construction of the Koch curve, the construction of Sierpinski’s triangle and his references to Mandelbrot’s and Barnsley’s texts.

There is an article in the 1993 *Physics Teacher*, by Talanquer and Irazoqui, that is found to be relevant to this study. Most of the article is irrelevant because it discusses the role fractals and chaotic systems have recently played in Chemistry. Important to this study is their explanation of the Koch curve, their definition of fractal and their generation of the equation for calculating fractal dimension. The authors tell how the Koch curve is generated and explain that the
Koch curve has infinite perimeter, yet encloses a finite area. Benoit Mandelbrot's definition of fractals is given as "a set of forms constructed normally by iteration and that are characterized by infinite detail, infinite length, no slope or derivative, fractional dimension, and self-similarity". The equation for fractal dimension is generated using cubes and the concept is well examined.

Through the internet, the researcher has found that the University of Pennsylvania offered a course in the Spring of 1995 called "Math 480: Introduction to Fractal Geometry". The course had prerequisites of basic set theory, including abstract notion of a function, finite versus, infinite sets, countable versus, uncountable sets, three semesters of calculus and linear algebra. The syllabus is also found at this site. This course was not instrumental in this study because the level was much too difficult for a community college based course, however, at this internet location there are computer programs which can be downloaded and used to create fractal images. Available were: Iterated Function System Demo Using Turtle Graphics and Generating Barnsley's Fern via the Chaos Game. Relevant to this study is the Iterated Function System Demo Using Turtle Graphics. The program uses a recursive algorithm and the encoded IFSs included the Sierpinski Gasket and the Koch Curve both of which are included in the proposed mini-course.
There are several books and texts that the researcher found useful in this study. The first is CHAOS: Making a New Science, by James Gleick. The text is very readable for non-mathematicians and people with no mathematical background could understand the basic theory behind chaos and fractals. There is a very thorough history of chaos and fractals which has been very instrumental in developing lesson one of this study. There is a chapter on the complex plane and the development of the Julia set and Mandelbrot set, which was helpful when creating lesson seven. Notions of strange attractors are discussed in detail by the author. Gleick also includes a section on iterations as they relate to the Koch snowflake and the Sierpinski triangle, which was useful when constructing lesson one. The ideas developed in this book and the way the author presents them were very valuable to the researcher.

Hans Lauwerier published a book entitled Fractals: Endlessly Repeated Geometric Figures which is devoted to fractals on a higher mathematical level than Gleick. The concepts of iteration, self-similarity, attractors, Newton’s Method, orbits, and fixed points are explained in mathematical detail. Appendix A is a review of complex numbers including graphing, operations on complex equations, compositions, an iterative process involving complex equations and an introduction to the Julia fractal. Appendix B is a compilation of over forty
The Fractal Geometry of Nature, written by Benoit Mandelbrot is of utmost importance to this study because Mandelbrot is generally considered to be the mathematician who invented the field of study called fractal geometry. The text is for the most part understandable to students with a knowledge base of one semester of calculus and operations on complex numbers and functions. Mandelbrot explains that nature is not accurately represented by Euclidean geometry, but instead is irregular and complex, using random variables in its shape and form. The length of a coastline, the rigidity of a mountain, the branching of a fern and a tree are all examples discussed by the author.

Mandelbrot also discusses iteration, which is imperative to this study. Mandelbrot relates iteration to the construction of the Koch curve and the Sierpinski triangle. Fractal dimension is discussed in relation to the length of a coastline. Mandelbrot also develops the Mandelbrot set, named after him, and explains its importance to fractal geometry.

A First Course in Chaotic Dynamical Systems is a text written by Robert L. Devaney for a college-based course on fractal geometry. The researcher found this text useful, though it goes well beyond the realm of this study. Sections related to this study explain iterative processes, types of orbits, graphical and
orbital analysis, fixed and periodic points and the properties of Newton's Method.

Also found in the text is the generation of the Sierpinski triangle and the Koch snowflake. Devaney thoroughly explains the Mandelbrot set with in-depth explanations of the attracting 2-cycle region and the attracting fixed point region.
CHAPTER 3

Procedure

Introduction

The purpose of this chapter is to explain the procedures the researcher used to construct the Mini Introduction to Fractal Course for community college students. The topics discussed in this chapter include the selection of related materials, the selection of related fractal topics and the construction of individual lesson plans for the Mini Introduction to Fractal Course.

Selection of Related Materials

The researcher selected materials for this study from several sources. All materials that deal with fractals are considered suitable material and reviewed by the researcher for the construction of the lesson plans. In this process the researcher included books related to fractal geometry, articles from magazines and journals and computer programming books that involve creating fractals with a computer. The researcher used Savitz Library at Rowan College of New Jersey, Richard Stockton College Library of New Jersey, Atlantic Community College Library of New Jersey and an Archie internet search as the primary sources for locating related material.
Selection of Related Fractal Topics

After the researcher had evaluated the related material, the decision had to be made which fractal topics should be included in the construction of the fractal lessons. The following guideline was used by the researcher to separate the topics that should be in the fractal lessons. The researcher used a typical student who has successfully completed Calculus-I at Atlantic Community College or a student who has similar mathematical experience in subject matter such as evaluating functions. Also, using a scientific calculator is considered typical for a community college student. Material that had the following criteria was included by the researcher for the lesson plans.

The decision was made that any necessary mathematics concepts that were needed for a fractal lesson were to be explained just before they were needed in the fractal lesson. This decision was made to keep the student continually interested in fractals throughout the course.

If the concepts were explained using a Calculus method, then the researcher looked for an alternative method to explain the same concepts without using the Calculus method. If such a method existed, then the concepts were included in the lessons.

The researcher made the decision that the first lesson should involve simply looking at fractal pictures and that the last lesson of the course should be
to create the Mandelbrot Set picture using a computer. Any mathematical and fractal concepts necessary to make the transition smooth from one concept to the other was included in the lesson plans of the course.

Construction of Individual Lesson Plans

The researcher constructed the individual lessons such that each lesson’s concepts could be explained in an hour and a half of class time. Each lesson plan included definitions and examples of them. The researcher made the decision to create an interest in fractals for the students in Lesson-1 without using any mathematical concepts of fractals. Lesson-1 was created using fractal pictures to help develop the student’s ability to see the similar shapes in different fractal pictures. The Koch Curve, Sierpinski’s Triangle, the Mandelbrot Set and Newton’s Approximation third degree polynomial pictures were used in Lesson-1 to accomplish this goal.

Lesson-2 was created to show how to create the Koch Curve using paper and pencil and then how to create Sierpinski’s Triangle. Concepts of areas and the lengths of these fractals were explained. Before the concept of iteration a concept of functions and mathematical operations on functions is needed, so a review of functions was introduced in Lesson-3 before iterations. Lesson-4 was to introduce iterations using a scientific calculator.
After the lesson on iteration (Lesson-4), Lesson-5 explains how to find an orbit of a seed and how to classify points. Both of these are applications of iteration.

Before creating fractals, an understanding of fractal dimension and how to calculate them are needed to study and create fractals. Therefore Lesson-6 was created to provide an introduction to the concept of fractal dimension and how to calculate fractal dimension and the similarities between the Euclidean and fractal dimensions.

To understand and create the Mandelbrot set, the concept of what a complex number is and how to perform mathematical operations on them are vital. Therefore Lesson-7 was created to explain complex numbers, how to perform mathematical operations on them and how to find the norm of a complex numbers.

To create the Mandelbrot set picture the concepts of what a complex plane is and how to plot complex numbers on the complex plane are important. Lesson-8 explains concepts such as how to plot complex numbers before creating the Mandelbrot set pictures.

Lesson-9 explains what the Mandelbrot set is and how to test a point which is in complex plane to see if it is in the Mandelbrot set or not in the
Mandelbrot set. This test is the last concept needed before creating the Mandelbrot set which is explained in Lesson-9.

Lesson-10 involves creating a Mandelbrot set picture using a computer which applies all the concepts that were explained in the previous lessons and concludes the mini introduction to fractal course. In Lesson-10, a QBASIC program code is given and the computer instruction will be explained to the students during the last class. This will conclude the course.
CHAPTER 4

Five Week Mini Course in Fractals

Introduction

This chapter is divided into three sections. They are the introduction, a short description of each lesson plan and a detailed description of what will take place in each lesson.

Description of Lesson Plans

A brief description of each lesson of the Mini Introduction to Fractal Course is explained below:

Lesson-1: Lesson-1 gives the students a brief history of fractals and self-similarities in the Koch Curve, Sierpinski's Triangle and the Mandelbrot set.

Lesson-2: Lesson-2 teaches the students how the Koch Curve and Sierpinski's Triangle are created and the mathematical relationship between the lengths and the areas of the objects.

Lesson-3: Lesson-3 is a review of functions and operations on functions.

Lesson-4: Lesson-4 explains what an iterative process is and includes experiments involving iteration using a scientific calculator.
Lesson-5: Lesson-5 explains about orbits and fixed points such as attracting and repelling fixed points.

Lesson-6: Lesson-6 explains about fractal dimension and how to calculate it.

Lesson-7: Lesson-7 explains what a complex number is and mathematical operations on complex numbers.

Lesson-8: Lesson-8 explains how to plot complex numbers on the complex plane and how to find the norm of a complex number.

Lesson-9: Lesson-9 explains what the Mandelbrot Set is and how to test whether a point is in the Mandelbrot set or outside of the Mandelbrot Set.

Lesson-10: Lesson-10 explains how to create the Mandelbrot Set using QBASIC computer language and how to use and create Mandelbrot Set pictures using Winfract fractal generator.

Lesson Plans

The purpose of this section is to explain each lesson plan in detail. This includes introducing, explaining and reviewing the material for each lesson.
Lesson Plan 1

Lesson 1: Fractal Pictures

Rationale: To understand self-similarity and apply self-similarity to fractals, this is the basic building block for fractal geometry. The students should achieve the ability to identify self-similar shapes in fractal pictures.

Goal: The students will be able to identify self-similar fractals in pictures such as the Koch Curve, Sierpinski’s Triangle, the Mandelbrot Set and Newton’s approximation pictures.

Objectives:

Cognitive: After the presentation of this class the students will do the following:

Identify self-similarities and create the Koch Curve, Sierpinski’s Triangle, Mandelbrot set pictures and Newton’s Approximation pictures with 70% accuracy.

Content:

1. A brief History of Fractals
2. Self-similarity in the Koch Curve.
4. Self-similarity in the Mandelbrot picture.
5. Self-similarity in Newton’s Approximation Pictures.

Activities:

Initiating: The teacher begins by welcoming the students, and he puts up pictures of the Koch Curve and Sierpinski’s Triangle and he asks the students to look at the pictures and asks if they see any similarities in the pictures.

Developing: Teacher will explain the following:

1. A Brief History of Fractals.

The history of fractals is not a long one. It began in 1975 with mathematician Benoît Mandelbrot’s revolutionary paper “A Theory of Fractal Sets”. Mandelbrot wrote “The Fractal Geometry of Nature” after “A Theory of Fractal Set”. Mathematicians such as Waclaw Sierpinski, David Hilbert, Georg Cantor and Helge von Koch have crafted fractals as abstract entertainment. These mathematicians’ contributions to fractals will be discussed in class.
2. Self-similar shapes in the Koch Curve.

Ask students if there is any similarity between Figure 1-1 and Figure 1-2. **Answer:** Yes, Figure 1-1 is a line segment, Figure 1-2, is made up of four similar line segments. Ask students if there is any similarity in Figure 1-2 and Figure 1-3. **Answer:** Yes, Figure 1-3 is made up of 4 objects similar to Figure 1-2.
3. **Self-similar shapes in Sierpinski's Triangle.**

The teacher will ask the students to find self-similar shapes in Sierpinski's Triangle.

**Figure 2 - 1**

Ask students if there is any similarity between Figure 2-1 and Figure 2-2.

**Answer:** Inside the initial triangle there are four similar triangles with the middle triangle shaded in black.

**Figure 2 - 2**

Ask students if there is any similarity between Figure 2-2 and Figure 2-3.

**Answer:** There are similar triangles. Each non-shaded triangle in Figure 2-2, contains four triangles and the middle triangle is shaded in blue in Figure 2-3.
**Answer:** There are similar triangles. Each non-shaded triangle in Figure 2-2, contains four triangles and the middle triangle is shaded in blue in Figure 2-3.

There are three objects similar to Figure 2-2 in Figure 2-3, if we ignore the colors.

![Figure 2 - 3](image)

Ask students if there is any similarity between Figure 2-3 and Figure 2-4.

**Answer:** There are similar triangles. Each non-shaded triangle (9 triangles) in figure 2-3 contains four triangles and the middle triangle is shaded in red in Figure 2-4. There are three objects similar to Figure 2-3, and nine objects similar to Figure 2-2, in Figure 2-4, if we ignore the colors.

![Figure 2 - 4](image)

Ask students if there is any similarity between Figure 2-4 and Figure 2-5.
**Answer:** There are similar triangles. Each non-shaded triangle (27 triangles) in Figure 2-4 contains four triangles and the middle triangle is shaded in yellow in Figure 2-5. There are three objects similar to Figure 2-4 and nine objects similar to Figure 2-3, and 27 objects similar to Figure 2-2 in Figure 2-5, if we ignore the colors.
4. Self-similarity in the Mandelbrot picture.

Ask students if there is any similarity between Figure 3-1 and Figure 3-2.

Answer: There are lots of objects that are similar to Figure 3-2 in Figure 3-1.
5. Self-similarity in Newton's Approximation Picture.

Ask students if there is any similarity between Figure 4-1 and Figure 4-2.

Answer: There are lots of objects that are similar to Figure 4-2 in Figure 4-1.
Culminating:

Brief review of similar shapes in the Koch Curve, Sierpinski's Triangle, the Mandelbrot set and Newton's Approximation pictures.

Exercise: Identify the self-similarity in the given picture.

Figure A1-1

Figure A1-2
Answer: There are lot of objects similar to Figure A1-2 in Figure A1-1.
Lesson Plan 2

Lesson 2: Iterations A Geometrical Approach

Rationale: To understand self-similarity and apply self-similarity to fractals, is the basic building block for fractal geometry. The students should achieve the ability to identify self-similar shapes in fractal pictures.

Goal: The students will create the Koch Curve and Sierpinski’s Triangle.

Objectives:

Cognitive: After the presentation of this class the students will do the following:

Create the Koch Curve and Sierpinski’s Triangle and make fractal pictures such as the Koch Curve, Sierpinski’s Triangle with 70% accuracy.

Content:

1. Creating the Koch Curve.

2. Creating Sierpinski’s Triangle.
**Activities:**

**Initiating:** The teacher begins by welcoming students, and puts up the pictures of the Koch Curve and Sierpinski's Triangle and asks the students to look at the pictures and asks that they look for any similarities in the pictures.

**Developing:** The teacher will explain the following:

Self-similar shapes in the Koch Curve by asking students to observe the shapes.

**Step-1:** Let $L$ be the length of the straight line. Figure 1-1.

![Figure 1-1](image)

**Step-2:** The line segment is divided into three equal parts and the middle part is replaced with an equilateral triangle without the base. This is called the
first iteration as shown in Figure 1-2. The length of the new object will be

\[ 4 \left( \frac{1}{3} \right) L \text{ or } L + \left( \frac{1}{3} \right) L. \text{ The length of the object is increased by } \left( \frac{1}{3} \right) L. \]

Step -3: Now each line segment in Figure 1-2 is divided into three equal parts and the middle part is replaced with an equilateral triangle without the base as shown in Figure 1-3. This is called the second iteration. The length of the new object will be

\[ 4 \left( \frac{4}{3} \right) \left( \frac{1}{3} \right) L. \text{ i.e. } 4^2 \left( \frac{1}{3} \right)^2 L \text{ or } \left( \frac{4}{3} \right)^2 L. \]

If each straight line segment of the previous object is divided into three equal parts and the middle part is replaced with an equilateral triangle without the base repeatedly, the curve created is known as the Koch Curve. If this process happens \( n \) times it is known as the \( n^{th} \) iteration of the Koch Curve. The length of the new object will be \( \left( \frac{4}{3} \right)^n L \), where \( n \) is the number of iterations.

Figure 1 - 3
2. **Self-similar shapes in Sierpinski’s Triangle.**

![Figure 2-1](image)

**Step 1:** Start with an equilateral Triangle as shown in Figure 2-1. Assume $L$ to be the length of each side and $A$ to be the area of the equilateral triangle.

![Figure 2-2](image)

**Step 2:** Connect the midpoints of the triangle and shade the equilateral triangle in the middle as shown in Figure 2-2. This is called the first iteration and the color used in the iteration is black. The original unshaded triangle contains four identical smaller equilateral triangles. The area of each of the small equilateral triangles will be $A/4$, since the area of the original equilateral triangle is $A$. The shaded area of the triangle will be $A/4$ and the unshaded area will be 3 times the shaded area or $3/4$ of the original equilateral triangle. If we cut out the
middle equilateral triangle the remaining area will be $\frac{3}{4}$ of the area that we started with or $A - \left(\frac{1}{4}\right)A$.

Figure 2 - 3

**Step 3:** Connect the midpoints of the unshaded equilateral triangles and shade the equilateral triangle in the middle of the unshaded triangles with blue color as shown in Figure 2-3. This is called the second iteration and the color used in the iteration is blue. Now we started with three unshaded triangles that have an area of $A/4$ and each unshaded triangle contains four smaller equilateral triangles each having an area of $\frac{1}{4}$ of $A/4$. If the blue region is cut out the area of the remaining object will be $A - \left(\frac{1}{4}\right)A - 3\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)A$.

Figure 2 - 4
Step 4: Repeat the same process with the remaining unshaded triangles and use the color red as shown in Figure 2-4. This is called the third iteration and the color used in the iteration is red. Now we have 3 times the number of unshaded triangles as the previous step (i.e. $3 \times 3 = 9$ equilateral triangles). If we cut out the red regions the remaining area of the object will be:

$$A = \left(1 - \left(\frac{1}{4}\right)^2\right)A - 3 \left(\frac{1}{4}\right)^2 A = A - \left(\frac{1}{4}\right)^2 A - 3 \left(\frac{1}{4}\right)^2 A$$

$$A = \frac{1}{4} A \left(1 + 3 \left(\frac{1}{4}\right) A + 3^2 \left(\frac{1}{4}\right)^2 A\right) = A \left(1 - \frac{1}{4} \left(1 + 3 \left(\frac{1}{4}\right) A + 3^2 \left(\frac{1}{4}\right)^2 A\right)\right)$$

![Figure 2-4](image)

Step 5: Repeat step 3 in the remaining unshaded triangles and use yellow color as shown in Figure 2-5. This is called the fourth iteration and the color used in the iteration is yellow. If we cut the yellow regions out, the area of the remaining object will be:

$$A = \frac{1}{4} A \left(1 - \frac{1}{4} \left(1 + 3 \left(\frac{1}{4}\right) A + 3^2 \left(\frac{1}{4}\right)^2 A\right)\right)$$

Step 6: Repeat step 2 through step 5 continuously.
In general find an unshaded equilateral triangle and connect the midpoints of the sides and shade the equilateral triangle in the middle of the original equilateral triangle. Continue the process with the rest of the unshaded triangles. If this process is done \( n \) times it is known as the \( n^{th} \) iteration of Sierpinski's Triangle. If the shaded area is cut out, the area of the remaining object will be as given below.

\[
A \left(1 - \frac{1}{4} \left(1 + \frac{3}{4} + \left[\frac{3^2}{4}\right]^2 + \left[\frac{3^3}{4}\right]^3 + \ldots + \left[\frac{3^n}{4}\right]^{n-1}\right)\right)
\]

Culminating:

1. Brief review of the Koch Curve, the Koch Curve length, Sierpinski's Triangle, and the area of Sierpinski's Triangle.

Exercise: Create a third iteration Koch Curve.
Lesson Plan 3

Lesson 3: Functions

Rationale: To understand fractals, a basic understanding of functions and operations on functions is necessary.

Goal: Evaluating mathematical functions and mathematical operations on functions such as addition, subtraction, multiplication, division and compositions of functions.

Objectives:

Cognitive: After the presentation of this class the students will do the following:

Students will be able to evaluate and perform mathematical operations on functions with 70% accuracy.

Content:

1. Evaluating Functions.
2. Mathematical operations on functions.
3. Composition of functions.

Activities:

Initiating: The teacher begins by welcoming students.

Developing: Teacher will explain the following:

1. How to evaluate functions.
2. How to add, subtract, multiply and divide functions.

3. How to find the composition of two functions.

Let \( f(x) = x^2 + x - 3 \) and \( g(x) = x - 1 \).

### 1. Evaluating Functions.

**Examples:**

1. \( f(2) = 2^2 + 2 - 3 = 4 + 2 - 3 = 3 \)

2. \( f(-1) = (-1)^2 + (-1) - 3 = 1 - 1 - 3 = -3 \)

3. \( f(a) = a^2 + a - 3 \)

4. \( f(x + h) = (x + h)^2 + (x + h) - 3 = x^2 + 2xh + h^2 + x + h - 3 \)

### 2. Mathematical operations on functions.

#### Adding Functions:

\[
(f + g)(x) = f(x) + g(x)
\]

**Example:** \( (f + g)(x) = f(x) + g(x) = (x^2 + x - 3) + (x - 1) = x^2 + 2x - 4 \)

#### Subtracting Functions:

\[
(f - g)(x) = f(x) - g(x)
\]

**Example:** \( (f - g)(x) = f(x) - g(x) = (x^2 + x - 3) - (x - 1) = x^2 - 2 \)

#### Multiplication of Functions:

\[
(fg)(x) = f(x)g(x)
\]
Example: \( (fg)(x) = f(x)g(x) = (x^3 + x - 3)(x - 1) = x^3 - 4x + 3 \)

**Division of Functions:**

\[
\left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)}, \text{ given } g(x) \neq 0
\]

**Example:**

\[
\left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} = \frac{x^4 + x - 3}{x - 1} = (x + 2) - \frac{1}{(x - 1)^2}, \text{ where } x \neq 1
\]

3. **Composition of Functions.**

\( f \circ g(x) = f(g(x)) \)

**Example 1:**

\( f \circ g(x) = f(g(x)) = (x - 1)^2 + (x - 1) - 3 = x^2 - 2x + 1 + x - 1 - 3 = x^2 - x - 3 \)

**Example 2:**

\( g \circ f(x) = g(f(x)) = (x^2 + x - 3) - 1 = x^2 + x - 4 \)

**Culminating:**

Brief review of the functions and mathematical operations on functions.

**Exercise:**

Let \( f(x) = x^2 + 3x - 2 \) and \( g(x) = x - 2 \).

**Evaluate the following:**

1. (a) \( f(2) \)  (b) \( g(2) \)  (c) \( f(a - 1) \)
2. (a) \( f(2) + g(-1) \)  \hspace{1cm} (b) \( f(a) - g(-a) \)

(c) \( f(x)g(x) \)  \hspace{1cm} (d) \( \frac{f(x)}{g(x)} \)

3. (a) \( f \circ g(x) \)  \hspace{1cm} (b) \( g \circ f(x) \)

(c) \( f \circ f(x) \)  \hspace{1cm} (d) \( g \circ g(x) \)

\textbf{Answers:}

1. (a) \( f(2) = 8 \)  \hspace{1cm} (b) \( g(2) = 0 \)

(c) \( f(a - 1) = a^2 + a - 4 \)

2. (a) \( f(2) + g(-1) = 5 \)  \hspace{1cm} (b) \( f(a) - g(-a) = a^2 + 4a \)

(c) \( f(x).g(x) = x^2 + x^2 - 8x + 4 \)

(d) \( \frac{f(x)}{g(x)} = x + 5 + \frac{8}{(x - 2)} \)

3. (a) \( f \circ g(x) = x^2 + x - 2 \)

(b) \( g \circ f(x) = x^2 + 3x - 4 \)

(c) \( f \circ f(x) = x^4 + 6x^4 + 5x^2 - 3x - 4 \)

(d) \( g \circ g(x) = x - 4 \)
Lesson Plan 4

Lesson 4: Iterations Using a Scientific Calculator

Rationale: To understand fractals, a basic understanding of iterations on functions is necessary.

Goal: Using a scientific calculator to find the iterations for a given function and a starting number.

Objectives:

Cognitive: After the presentation of this class the students will do the following:

Students will be able to calculate mathematical iterations with 70% accuracy.

Content: Iterations Using a Scientific Calculator.

Activities:

Initiating: The teacher begins by welcoming students.

Developing: Teacher will explain the following:

How to calculate iterations using a scientific calculator.
If an instruction or a set of instructions is executed continuously it is known as iteration.

**Example 1:**

**Step 1:** Take the number 100 and find the square root. Record the step number and the answer of the step.

**Step 2:** Use the answer of the previous step and find the square root.

<table>
<thead>
<tr>
<th>Step Number</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>1</td>
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</tr>
<tr>
<td>2</td>
<td>3.162277</td>
</tr>
<tr>
<td>3</td>
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<td>4</td>
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</tr>
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<td>7</td>
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</tr>
<tr>
<td>8</td>
<td>1.004807</td>
</tr>
</tbody>
</table>

**Example 2:**

**Step 1:** Take the number 0.001 and find the square root. Record the step number and the answer of the step.

**Step 2:** Use the answer of the previous step and find the square root.
<table>
<thead>
<tr>
<th>Step Number</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.001</td>
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<tr>
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<td>2</td>
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<tr>
<td>10</td>
<td>0.993277</td>
</tr>
</tbody>
</table>

In the above examples: The square root is the function and the step number is the number of iterations. (i.e. Example: Step 3 is the third iteration and the n\textsuperscript{th} Step is the n\textsuperscript{th} iteration).

Note: When an iteration increases the answers in both examples approach 1.

**Culminating:**

Brief review of iteration using a scientific calculator.

**Exercise:**

1. Start with the number 1.1 and use the square as the function. Find up to 10 iterations.

2. Start with the number 0.9 and use the square as the function. Find up to 10 iterations.
Answers:

1.

<table>
<thead>
<tr>
<th>Step Number</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.1</td>
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<tr>
<td>1</td>
<td>1.21</td>
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<td>1.4641</td>
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<tr>
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</tr>
<tr>
<td>10</td>
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</tr>
</tbody>
</table>

Note that when the iteration increases the answer increases. (i.e. The answer approaches infinity).

2.

<table>
<thead>
<tr>
<th>Step Number</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
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<td>3.73E-24</td>
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<tr>
<td>10</td>
<td>1.39E-47</td>
</tr>
</tbody>
</table>

Note that when the iteration increases the answer decreases. (i.e. The answer approaches zero).
Lesson Plan 5

Lesson 5: Orbits and Fixed Points.

Rationale: To understand fractals, an understanding of orbits and fixed points is necessary.

Goal: To understand orbits and the behavior of fixed points.

Objectives:

Cognitive: After the presentation of this class the students will do the following:

1. Students will be able to find orbits with 70% accuracy.
2. Classify whether a point is an attracting or a repelling fixed point by inspecting the orbits with 70% accuracy.

Content:

1. Orbits.
2. Fixed points.
3. Attracting Fixed Points.
4. Repelling fixed Points.

Activities:

Initiating: The teacher begins by welcoming students.
Developing: Teacher will explain the following:

1. Orbits
2. Fixed Points
3. Repelling fixed points.
4. Attracting fixed points.

1. Orbits:

Definition: Given \( x_0 \in \mathbb{R} \), we define the orbit of \( x_0 \) under \( F \) to be the sequence of points \( x_0, x_1 = F(x_0), x_2 = F^2(x_0), ... x_n = F^n(x_0), ... \) The point \( x_0 \) is called the seed of the orbit.

This means simply to apply the same process to the previous results. In the following example we are going to use square root function \( F(x) = \sqrt{x} \). The first point is the \( x_1 = \sqrt{x_0} \), \( x_2 = \sqrt{\sqrt{x_0}} = \sqrt{x_1} \) and so on.

Example:

Let's Find the orbit of \( Z_0 = 25 \), \( Z_0 = 100 \) and \( Z_0 = 1000 \), in the function

\[
Z_{n+1} = \sqrt{Z_n}.
\]
The orbit of the seed 25 of the function $F(x) = \sqrt{x}$ is 5, 2.2360, 1.4953, 1.2228 ... and so on. The orbit of the seed for 100 of the function $F(x) = \sqrt{x}$ is 10, 3.1627, 1.7782, 1.3353, 1.1547 ... and so on as shown in Table 5-1.

Note: For $Z_0 = 25$, $Z_0 = 100$ and $Z_0 = 1000$, $Z_n \to 1$ when $n \to \infty$.

2. Fixed Points:

A fixed point is a point $x_*$ that satisfies $F^n(x_0) = x_*$ (i.e. $x_*$, $x_1 = F(x_0) = x_*, x_2 = F^2(x_0) = x_*, ..., x_n = F^n(x_0) = x_*$, ...) The orbit of a fixed point is the constant sequence. A fixed point never moves for any number of iterations.

Table 5-1

<table>
<thead>
<tr>
<th>Orbit Number (Z)</th>
<th>Seed (Z)</th>
<th>Seed (Z)</th>
<th>Seed (Z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>25</td>
<td>100</td>
<td>1000</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>10</td>
<td>4.47271</td>
</tr>
<tr>
<td>2</td>
<td>3.87298</td>
<td>1.77828</td>
<td>1.33533</td>
</tr>
<tr>
<td>3</td>
<td>1.49532</td>
<td>1.77828</td>
<td>1.33533</td>
</tr>
<tr>
<td>4</td>
<td>1.22284</td>
<td>1.77828</td>
<td>1.33533</td>
</tr>
<tr>
<td>5</td>
<td>1.10582</td>
<td>1.77828</td>
<td>1.33533</td>
</tr>
<tr>
<td>6</td>
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<td>1.77828</td>
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</tr>
<tr>
<td>7</td>
<td>1.00003</td>
<td>1.77828</td>
<td>1.33533</td>
</tr>
<tr>
<td>8</td>
<td>1.00000</td>
<td>1.77828</td>
<td>1.33533</td>
</tr>
<tr>
<td>9</td>
<td>1.00000</td>
<td>1.77828</td>
<td>1.33533</td>
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<tr>
<td>10</td>
<td>1.00000</td>
<td>1.77828</td>
<td>1.33533</td>
</tr>
<tr>
<td>11</td>
<td>1.00000</td>
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<td>1.33533</td>
</tr>
<tr>
<td>12</td>
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<td>1.33533</td>
</tr>
<tr>
<td>15</td>
<td>1.00000</td>
<td>1.77828</td>
<td>1.33533</td>
</tr>
</tbody>
</table>
Example:
For function $F(x) = \sqrt{x}$, $x = 0$ and $x = 1$ are fixed points. (i.e. when $n \rightarrow \infty$.

$Z_n \rightarrow x_n$

<table>
<thead>
<tr>
<th>Orbit Number (Z)</th>
<th>Seed</th>
<th>Seed</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 5-2

The orbit of the seed 1 for the function $F(x) = \sqrt{x}$ is 1, 1, 1, 1 ... and so on. (i.e. $F^n(x_0) = x_0$ is true). The orbit of the seed 0 for the function $F(x) = \sqrt{x}$ is 0, 0, 0, 0 ... and so on. (i.e. $F^n(x_0) = x_0$ is true). Seeds 1 and 0 are fixed points, since $F^n(x_0) = x_0$ is true for the seeds 1 and 0. See Table 5-2.

3. Repelling Fixed Points:

Example:

For function $F(x) = x^3$ when $x > 1$, the orbits are going away from the fixed point $x = 1$ and they approach infinity as shown in the Table 5-3. When $x < -1$, the orbits are going away from the fixed point $x = -1$ and they approach
infinity as shown in the table below. The points \( x = 1 \) and \( x = -1 \) are called repelling fixed points for the function.

<table>
<thead>
<tr>
<th>Orbit Number (Z)</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<tr>
<td>1</td>
<td>1.21</td>
</tr>
<tr>
<td>2</td>
<td>1.4641</td>
</tr>
<tr>
<td>3</td>
<td>2.143589</td>
</tr>
<tr>
<td>4</td>
<td>4.594973</td>
</tr>
<tr>
<td>5</td>
<td>21.11378</td>
</tr>
<tr>
<td>6</td>
<td>445.7916</td>
</tr>
<tr>
<td>7</td>
<td>128730.1</td>
</tr>
<tr>
<td>8</td>
<td>3.05E+10</td>
</tr>
</tbody>
</table>

Table 5-3

Note: For \( Z_0 = 1.1 \), \( Z_0 = 2.1 \) and \( Z_0 = 3.1 \), the function \( Z_{n+1} = Z_n^2 \) approaches infinity.

For all \( |Z_0| > 1 \), the function \( Z_{n+1} = Z_n^2 \) approaches infinity. The point \( Z_0 = 1 \) is called a repelling point for the function \( Z_{n+1} = Z_n^2 \).

<table>
<thead>
<tr>
<th>Orbit Number (Z)</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
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<tr>
<td>2</td>
<td>0.6561</td>
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<tr>
<td>3</td>
<td>0.430487</td>
</tr>
<tr>
<td>4</td>
<td>0.185302</td>
</tr>
<tr>
<td>5</td>
<td>0.034337</td>
</tr>
<tr>
<td>6</td>
<td>0.001179</td>
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<tr>
<td>7</td>
<td>1.39E-06</td>
</tr>
<tr>
<td>8</td>
<td>1.33E-12</td>
</tr>
</tbody>
</table>

Table 5-4
Note: For $Z_0 = 0.9$, $Z_0 = 0.1$ and $Z_0 = 0.0001$, $Z_{n+1} = Z_n^2$ approaches zero.

For all $|Z_0| < 1$, the function $Z_{n+1} = Z_n^2$ approaches zero. The point at $Z_0 = 1$ is called a repelling point for the function $Z_{n+1} = Z_n^2$.

Note that for $Z_0 = 1$, $Z_{n+1} = 1$ for any number of iterations, when $Z_0 = 0$, $Z_{n+1} = 0$ for any number of iterations.

4. Attracting Fixed Points:

Example:

For the function $F(x) = x^2$, when $0 < x < 1$ the orbits are going toward the fixed point $x = 0$, and also when $-1 < x < 0$ the orbits are going toward the fixed point $x = 0$ as shown in the Table 5-4. The point $x = 0$ is called an attracting fixed point for the function.

Example:

Let's find the orbit of $Z_0 = 0.9$, $Z_0 = 0.1$ and $Z_0 = 0.0001$ in the function $Z_{n+1} = \sqrt{Z_n}$.
Table 5-5

<table>
<thead>
<tr>
<th>Orbit Number (Z)</th>
<th>Seed</th>
<th>Seed</th>
<th>Seed</th>
</tr>
</thead>
<tbody>
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<td>0.9</td>
<td>0.1</td>
<td>0.0001</td>
</tr>
<tr>
<td>1</td>
<td>0.948683</td>
<td>0.316228</td>
<td>0.01</td>
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<td>2</td>
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<td>0.1</td>
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<tr>
<td>3</td>
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<tr>
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<td>0.993437</td>
<td>0.865964</td>
<td>0.01</td>
</tr>
<tr>
<td>5</td>
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<td>6</td>
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<td>0.964662</td>
<td>0.865964</td>
</tr>
<tr>
<td>7</td>
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<td>0.982172</td>
<td>0.930572</td>
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<tr>
<td>8</td>
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<td>0.991046</td>
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<tr>
<td>9</td>
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<td>0.995513</td>
<td>0.982172</td>
</tr>
<tr>
<td>10</td>
<td>0.999897</td>
<td>0.997754</td>
<td>0.991046</td>
</tr>
<tr>
<td>11</td>
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</tr>
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</tr>
<tr>
<td>13</td>
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<td>0.999719</td>
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<tr>
<td>14</td>
<td>0.999994</td>
<td>0.999859</td>
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<tr>
<td>15</td>
<td>0.999997</td>
<td>0.999938</td>
<td>0.999719</td>
</tr>
</tbody>
</table>

For $Z_0 = 0.9$, $Z_0 = 0.1$ and $Z_0 = 0.0001$, $Z_{n+1} = \sqrt{Z_n}$ approaches 1.

Note that for $Z_0 = 1$, $Z_{n+1} = 1$ for any number of iterations. When $Z_0 = 0$,

$Z_{n+1} = 0$ for any number of iterations.

Note: For $Z_0 = 0.9$, $Z_0 = 0.1$ and $Z_0 = 0.0001$ the $Z_{n+1} = \sqrt{Z_n}$ approaches 1.

For all $0 < Z_0 < 1$ [i.e. $Z_0 \neq 1$, $Z_0 \neq 0$], the function $Z_{n+1} = \sqrt{Z_n}$ approaches 1.

The point 1 is called an attraction point for the function

$Z_{n+1} = \sqrt{Z_n}$. 
Culminating:

Brief review of iteration using a scientific calculator.

Exercise:

1. Find the orbits of the following seeds for the function \( F(x) = x^4 \)

\[ x_0 = 0.1, \ x_n = -0.1, \ x_n = 1.1, \ x_n = -1.1 \]

2. Find the fixed points of the function \( F(x) = x^4 \) and classify them as repelling fixed points or attracting fixed points.

Answers:

1.

<table>
<thead>
<tr>
<th>Orbit Number (n)</th>
<th>Seed</th>
<th>Seed</th>
<th>Seed</th>
<th>Seed</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1</td>
<td>1.1</td>
<td>-0.1</td>
<td>-1.1</td>
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<tr>
<td>1</td>
<td>0.0001</td>
<td>1.4641</td>
<td>0.0001</td>
<td>1.4641</td>
</tr>
<tr>
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<td>1E-08</td>
<td>4.554973</td>
</tr>
<tr>
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<td>1E-16</td>
<td>445.7916</td>
<td>1E-16</td>
<td>445.7916</td>
</tr>
<tr>
<td>4</td>
<td>1E-32</td>
<td>3.95E+10</td>
<td>1E-32</td>
<td>3.95E+10</td>
</tr>
<tr>
<td>5</td>
<td>1E-64</td>
<td>2.43E+42</td>
<td>1E-64</td>
<td>2.43E+42</td>
</tr>
</tbody>
</table>

2. Fixed points are \( x = -1, x = 0, x = 1 \). Repelling fixed points are \( x = -1, x = 1 \). An attracting fixed point is \( x = 0 \).
Lesson Plan 6

Lesson 6: Fractal Dimensions.

Rationale: Fractals cannot be studied without investigating their dimensions, because the formal definition of a fractal is given in terms of dimensions.

Goal: To understand fractal dimensions.

Objectives:

Cognitive: After the presentation of this class the students will do the following:

1. Calculate the dimension of a fractal with 70% accuracy.

Content:

1. Euclidean Dimensions.
2. Applying the general formula for Dimensions to verify Euclidean Dimensions
3. Fractal Dimensions

Activities:

Initiating: The teacher begins by welcoming students,

Developing: Teacher will explain the following:
1. Euclidean Dimensions.

2. Applying the general formula for Dimensions to verify Euclidean Dimensions

3. Fractal Dimensions

1. Euclidean Dimensions:

In Euclidean geometry, mathematical objects have the dimension of 0, 1, 2 and 3. Points have a Euclidean Dimension of 0. Lines have a Euclidean Dimension of 1. Squares have a Euclidean Dimension of 2. Cubes have a Euclidean Dimension of 3.

2. Applying the general formula for Dimension to verify Euclidean Dimension 1

Consider a line segment of unit length and divide it into four equal parts as shown in Figure 6-1. The line could be reconstructed by using four of the small parts of the unit length of the line segment. (i.e. Magnification factor 1/4 and number of components 4.) This can be verified in the dimension formula.
Consider a square which has unit length of one side and divide it into four equal parts as shown in Figure 6-2. The square could be reconstructed by using 16 of the small parts of the unit square. (i.e. The magnification factor is $1/4$ and number of components 16.) This can be verified in the dimension formula.

$$Nr^D = 1, \ N=16, \ r=1/4, \ D=2$$

$$Nr^D = 1 \Rightarrow (16)\left(\frac{1}{4}\right)^2 = 1$$
Consider a cube which has an unit length of each side and divide it into four equal parts as shown in Figure 6-3. The cube could be reconstructed by using 64 of the small parts of the unit cube. (i.e. The magnification factor is 1/4 and number of components 64.) This can be verified in the dimension formula.

\[ N r^D = 1, \quad N = 64, \quad r = 1/4, \quad D = 3 \]

\[ N r^D = 1 \Rightarrow (64) \left( \frac{1}{4} \right)^3 = 1 \]
3. Fractal Dimensions:

\[ N r^D = 1 \]
\[ N = \frac{1}{r^D} \]
\[ N = \left(\frac{1}{r}\right)^D \]
\[ D = \frac{\log(N)}{\log\left(\frac{1}{r}\right)} \]

**Example 1:** Koch Curve.

![Figure 6-4](image1)

\( r = 1/3 \) and \( N = 4 \) or \( 1/r = 3 \) and \( N = 4 \)

\[ D = \frac{\log(4)}{\log(3)} = \frac{\log(4)}{\log(3)} = 1.2618 \]

**Example 2:**

\( r = 1/9 \) (i.e. \( 1/r = 9 \)) \( N = 16 \)

![Figure 6-5](image2)
Culminating:

1. Brief review on fractal dimension.

Exercise:

1. Find the fractal dimension of the Sierpinski Triangle.

Answer:

\[
D = \frac{\log(N)}{\log(\sqrt{3})} = \frac{\log(4)}{\log(3)} = \frac{\log(4)}{\log(3)} = 1.2618
\]

The base of the triangle in Figure 6A-1 is 1/2 of the base of the triangle in Figure 6A-2. Therefore the \( r = 1/2 \). Three objects in Figure 6A-1 are needed to create the object in Figure 6A-2. Therefore \( N = 3 \).

\[
D = \frac{\log(N)}{\log(\sqrt{3})} = \frac{\log(3)}{\log(2)} = 1.584
\]
Lesson Plan 7

Lesson 7: Introduction To Complex Numbers

**Rationale:** To understand fractal geometry, performing operations on complex numbers is necessary.

**Goal:** To perform mathematical operations on complex numbers.

**Objectives:**

**Cognitive:** After the presentation of this class the students will do the following:

1. Students will be able to perform mathematical operations on complex numbers with 70% accuracy.

**Content:**

1. What is a complex number?
2. Adding and subtracting complex numbers.
3. Multiplying complex numbers.
4. Finding Conjugates of complex numbers.
5. Dividing complex numbers.
6. Finding the norm of a complex number.
Activities:

Initiating: The teacher begins by welcoming students.

Developing: Teacher will explain the following:

1. What is a complex number?

Assume that a number exits such that \( i^2 = -1 \). Where \( i \) is known as an imaginary number. A complex number is defined as \( a + bi \), where \( a \) and \( b \) are real numbers.

Examples:

1. \( Z_1 = 2 + 3i \)  
2. \( Z_2 = -1 + 4i \)

2. Adding and subtracting complex numbers.

Let \( Z_1 = a_1 + b_1i \) and \( Z_2 = a_2 + b_2i \).

Adding Complex Numbers:

\[
Z_1 + Z_2 = (a_1 + b_1i) + (a_2 + b_2i) = (a_1 + a_2) + (b_1 + b_2)i
\]

Examples: Let \( Z_1 = 2 + 3i \) and \( Z_2 = -1 + 4i \) then

\[
Z_1 + Z_2 = (2 + 3i) + (-1 + 4i) = (2 + (-1)) + (3 + 4)i = 1 + 7i
\]

Subtracting Complex Numbers:

\[
Z_1 - Z_2 = (a_1 + b_1i) - (a_2 + b_2i) = (a_1 - a_2) + (b_1 - b_2)i
\]

Example: Let \( Z_1 = 2 + 3i \) and \( Z_2 = -1 + 4i \) then
\[
Z_1 - Z_2 = (2 + 3i) - (-1 + 4i) = (2 - (-1)) + (3 - 4)i = 3 - i
\]

3. Multiplying Complex Numbers:

**Multiplication:**

\[
Z_1Z_2 = (a_1 + b_1i)(a_2 + b_2i) = (a_1a_2 + a_1b_2i + a_2b_1i + b_1b_2i^2)
= (a_1a_2 + (a_1b_2 + a_2b_1)i + b_1b_2(-1)) = (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i
\]

**Example:** Let \(Z_1 = 2 + 3i\) and \(Z_2 = -1 + 4i\) then

\[
Z_1Z_2 = (2 + 3i)(-1 + 4i) = (2(-1) + 2(4)i + 3(-1)i + 3(4)i^2)
= (-2 + 8i - 3i + 12(-1)) = -2 - 12 + 5i = -14 + 5i
\]

4. Finding Conjugates of a complex number:

**Complex Conjugates:** Let \(Z_1 = a_1 + b_1i\) then the conjugate of \(Z_1\) will be

\[
\overline{Z_1} = a_1 - b_1i
\]

**Example:** Let \(Z_1 = 2 + 3i\) and \(Z_2 = -1 + 4i\) then

The complex conjugate of \(Z_1\) is \(\overline{Z_1} = 2 - 3i\)

The complex conjugate of \(Z_2\) is \(\overline{Z_2} = -1 - 4i\).
5. Multiplication of a complex number by its conjugate:

\[ Z_1 \overline{Z}_1 = (a_1 + b_1i)(a_1 - b_1i) = (a_1a_1 - b_1(-b_1)i + a_1b_1i + b_1(-b_1)i^2) \]
\[ = (a_1^2 + a_1b_1i - b_1^2(-1)) = a_1^2 + b_1^2 \]

**Example:** Let \( Z_1 = 2 + 3i \) and \( Z_2 = -1 + 4i \) then multiplication of a complex number by its complex conjugate will be:

\[ Z_1 \overline{Z}_1 = (2 + 3i)(2 - 3i) = (2(2) + 2(-3)i + 2(3)i + 3(-3)i^2) \]
\[ = (4 + (-6 + 6)i - 9(-1)) = 4 + 9 = 13 \]

6. Finding the norm of a complex number:

**The Norm of a complex number:**

If \( Z_i = a_i + b_i i \) then the norm of \( Z_i \) is \( |Z_i| = \sqrt{a_i^2 + b_i^2} \).

**Example:** If \( Z_i = 2 + 3i \) then the norm of \( Z_i \) is

\[ |Z_i| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13} \]

Culminating:

Brief review of complex numbers and mathematical operations on complex numbers.
Exercise:

Homework Assignments for Students:
Let $Z_1 = 1 + 5i$, $Z_2 = 12 + 5i$, $Z_3 = 3 + 4i$, $Z_4 = -1 + i$ and $Z_5 = 5 - 3i$

Perform the following mathematical operations:

1. (a) $Z_1 + Z_2$  (b) $Z_2 + Z_2$  (c) $Z_3 + Z_5$  (d) $Z_1 + Z_4$
2. (a) $Z_1 - Z_2$  (b) $Z_3 - Z_2$  (c) $Z_3 - Z_5$  (d) $Z_1 - Z_4$
3. (a) $Z_1 Z_2$  (b) $Z_2 Z_2$  (c) $Z_2 Z_5$  (d) $Z_1 Z_4$
4. (a) $\overline{Z_1}$  (b) $\overline{Z_3}$  (c) $\overline{Z_5}$  (d) $\overline{Z_4}$
5. (a) $Z_1 \overline{Z_1}$  (b) $Z_3 \overline{Z_3}$  (c) $Z_2 \overline{Z_5}$  (d) $Z_4 \overline{Z_4}$

Answers:

1. (a) $13 + 10i$  (b) $15 + 9i$  (c) $8 + i$  (d) $6i$
2. (a) $-11$  (b) $-9 - i$  (c) $-2 + 7i$  (d) $2 + 4i$
3. (a) $-13 + 6.5i$  (b) $16 + 63i$  (c) $2 + 11i$  (d) $-6 - 4i$
4. (a) $1 - 5i$  (b) $3 - 4i$  (c) $5 + 3i$  (d) $-1 - i$
5. (a) $26$  (b) $25$  (c) $34$  (d) $2$
Lesson Plan 8

Lesson 8: Plotting Complex Numbers.

Rationale: To understand fractal geometry, plotting complex numbers on the complex plane is necessary.

Goal: To plot complex numbers on the complex plane.

Objectives:

Cognitive: After the presentation of this class the students will do the following:

1. Students will be able to plot complex numbers on the complex plane with 70% accuracy.

Content:

1. The Complex Plane and plotting complex numbers.

2. Norm of a complex number.

Activities:

Initiating: The teacher begins by welcoming students.

Developing: Teacher will explain the following:

1. The Complex Plane and plotting complex numbers.

The Complex plane is similar to the rectangular coordinate system, but the X-axis is considered as the real axis and Y-axis is considered the imaginary axis.
complex number $Z = a + bi$, is the point which is $a$ units along the x-axis and $b$ units along the imaginary axis or i-axis as shown in the figure below.

![Diagram of complex number](image)

2. **Norm of a complex number**

The norm of a complex number is the distance from the origin to the complex number. The distance from the origin to a complex number $Z = a + bi$ will be $\sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}$.

**Example:**

Let $Z_1 = 2 + 3i$ and $Z_2 = -1 + 4i$

Find the norm of (a) $|Z_1| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$

(b) $|Z_2| = \sqrt{(-1)^2 + 4^2} = \sqrt{1 + 16} = \sqrt{17}$
Culminating:

Brief review on the complex plane and how to plot complex numbers on the complex plane.

Exercise:

Homework Assignments for Students:

Plot the following complex numbers on the complex plane and find the norms of each complex number.

1. \( Z_1 = 1 + 5i \)
2. \( Z_2 = 3 + 4i \)
3. \( Z_3 = -1 + i \)
4. \( Z_4 = 5 - 3i \)
Answers:

1. $|Z_1| = \sqrt{1^2 + 5^2} = \sqrt{1 + 25} = \sqrt{26}$

2. $|Z_2| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

3. $|Z_3| = \sqrt{(-1)^2 + 1^2} = \sqrt{1 + 1} = \sqrt{2}$

4. $|Z_4| = \sqrt{5^2 + (-3)^2} = \sqrt{25 + 9} = \sqrt{34}$
Lesson Plan 9

Lesson 9: Mandelbrot Set.

Rationale: To understand fractals, an understanding of orbits and fixed points is necessary.

Goal: To understand orbits and the behavior of fixed points.

Objectives:

Cognitive: After the presentation of this class the students will do the following:

1. Students will be able to find out whether an orbit is escaping or not, for a given point in the Mandelbrot set with 70% accuracy.

Content: 1. Escaping Orbits and points in the Mandelbrot Set.

2. Non-escaping Orbits and points not in the Mandelbrot Set.

Activities:

Initiating: The teacher begins by welcoming students.

Developing: Teacher will explain the following:

1. Escaping Orbits and points in the Mandelbrot Set.

2. Non-Escaping Orbits and points not in the Mandelbrot Set.
The Mandelbrot set is a collection of points in the complex plane. A point is in the Mandelbrot set if the orbit of the point is in the circle of radius 2 with the center at the origin. If the orbit of the point is outside the circle with the radius 2 and the center is at the origin then it is not in the Mandelbrot set.

1. Escaping Orbits and points in the Mandelbrot Set.

If the orbit of the seed \( Z_1 \) is \( Z_1, Z_2, Z_3, \ldots Z_n \) for a function

\[
Z_{n+1} = Z_n^2 + C
\]

and if the norm \( Z_n \) is greater than 2 for some \( Z_n \), then the orbit of \( Z_1 \) is escaping from Mandelbrot set. And therefore the seed \( Z_1 \) is not in the Mandelbrot set as shown in Figure 1-1.

---

Figure 1-1
Let’s find the orbit of $Z_0 = 0 + 0i$, $C = 0.38 + 0.4i$ for the function $Z_{n+1} = Z_n^2 + C$. The sequence of the points is given in Table 9-1. After $Z_{10}$, the orbit is out of the circle with the radius 2 with the center at the origin. It is said that the point $C = 0.38 + 0.4i$ is not in the Mandelbrot set, because the orbit of the point is escaping from the circle after the 10th iteration.

$Z_0 = 0 + 0i$

$Z_1 = Z_0^2 + C$

$Z_2 = Z_1^2 + C$

..................

$Z_{n+1} = Z_n^2 + C$

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<thead>
<tr>
<th>Sequence (n)</th>
<th>Real</th>
<th>Imaginary</th>
<th>Norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>C (Seed)</td>
<td>0.38</td>
<td>0.4</td>
<td>0.551725</td>
</tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>0.4</td>
<td>0.551725</td>
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<tr>
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<td>3</td>
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<td>0.913076</td>
<td>0.9132</td>
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<tr>
<td>4</td>
<td>-0.453</td>
<td>0.431357</td>
<td>0.6256</td>
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<tr>
<td>5</td>
<td>0.400</td>
<td>0.008835</td>
<td>0.3396</td>
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<tr>
<td>6</td>
<td>0.540</td>
<td>0.407060</td>
<td>0.6759</td>
</tr>
<tr>
<td>7</td>
<td>0.505</td>
<td>0.839244</td>
<td>0.9797</td>
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<td>1.2575</td>
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<td>9</td>
<td>-1.191</td>
<td>0.227004</td>
<td>1.2125</td>
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<td>10</td>
<td>1.747</td>
<td>-0.140809</td>
<td>1.7531</td>
</tr>
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<td>11</td>
<td>3.414</td>
<td>-0.052098</td>
<td>3.4146</td>
</tr>
<tr>
<td>12</td>
<td>12.0241</td>
<td>-0.228768</td>
<td>12.0262</td>
</tr>
</tbody>
</table>

Table 9-1
2. Non-Escaping Orbits and points not in the Mandelbrot Set.

If the orbit of the seed $Z_1$ is $Z_1, Z_2, Z_3, ..., Z_n$ for a function $Z_{n+1} = Z_n^2 + C$ and the norm of $Z_n$ is less than 2 for some $Z_n$ where $n$ is a very large integer then the orbit of $Z_1$ is a non-escaping orbit. Therefore the seed $Z_1$ is in the Mandelbrot set as shown Figure 1-2.
For the seed of $C = 0.36 + 0.21i$, after 17 iterations the orbit still did not escape as shown in Table 9-2. We assume that $C = 0.36 + 0.21i$ has a non-escaping orbit.

<table>
<thead>
<tr>
<th>Sequence (Z)</th>
<th>Real</th>
<th>Imaginary</th>
<th>Norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>C (Seed)</td>
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<td>0.21</td>
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<td>0</td>
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<td>0.21</td>
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<td>2</td>
<td>0.448</td>
<td>0.361200</td>
<td>0.5735</td>
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<tr>
<td>3</td>
<td>0.428</td>
<td>0.531829</td>
<td>0.6627</td>
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<tr>
<td>4</td>
<td>0.260</td>
<td>0.665251</td>
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<td>5</td>
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<td>0.556391</td>
<td>0.5596</td>
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<td>0.183554</td>
<td>0.2001</td>
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<td>0.229806</td>
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<td>0.5367</td>
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<tr>
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<td>0.483171</td>
<td>0.8435</td>
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<td>0.8110</td>
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<tr>
<td>11</td>
<td>0.061</td>
<td>0.564803</td>
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</tr>
<tr>
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<td>0.045</td>
<td>0.279398</td>
<td>0.2830</td>
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<td>13</td>
<td>0.284</td>
<td>0.239018</td>
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<td>14</td>
<td>0.385</td>
<td>0.343468</td>
<td>0.5162</td>
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<td>15</td>
<td>0.391</td>
<td>0.474734</td>
<td>0.6147</td>
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<td>16</td>
<td>0.267</td>
<td>0.560822</td>
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<tr>
<td>17</td>
<td>0.105</td>
<td>0.543582</td>
<td>0.5537</td>
</tr>
</tbody>
</table>

Table 9-2

Culminating:

Brief review of escaping orbits, non-escaping orbits and Mandelbrot set points.
Exercise:

1. Find whether the point $C = 0.54 + 0.42i$ is in the Mandelbrot set or not in the Mandelbrot Set (Maximum iteration 12)

Answers:

1. The point $C = 0.54 + 0.42i$ is not in the Mandelbrot Set, because after the 4th iteration, the norm of the point is greater than 2.

<table>
<thead>
<tr>
<th>Sequence ($z$)</th>
<th>Real</th>
<th>Imaginary</th>
<th>Norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>C (Seed)</td>
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<td>0.42</td>
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<td>4</td>
<td>-1.865</td>
<td>1.065028</td>
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</table>
Lesson Plan 10

Lesson 10: Mandelbrot Set Computer Experiments

Rationale: To understand fractals by seeing fractal pictures provides a better understanding than just imagining them. Modern computers are one of the best tools with which to see fractals.

Goal: To create the Mandelbrot set picture using computers.

Objectives:

Cognitive: After the presentation of this class the students will do the following:

Students will be able to create the Mandelbrot set using two colors with 70% accuracy.

          2. Creating and exploring the Mandelbrot Set using WINFRACT fractal generator.

Activities:

Initiating: The teacher begins by welcoming students.

Developing: Teacher will provide a QBASIC program to create the Mandelbrot Set. The program code will be given to the students, the teacher
will explain the program code and the students will be asked to use it to create the two color Mandelbrot Set.

10 REM ****** MANDELBROT SET ******
20 REM ****** TWO COLORS ******
30 SCREEN 3 : CLS
40 WINDOW (-2.2, -1.4)-(1.1,1.4)
50 N1=160 : N2=100
60 FOR I=-N1 TO N1 : A--0.55+I*N1
70 FOR J=0 TO N2 : B=1.4*J/N2
80 U=4*(A*A+B*B) : V=U-2*A+1/4
90 IF U+8*A+15/4<0 THEN K=1 : GOTO 170
100 IF V-SQR(V)+2^A-1/2<0 THEN K=1 : GOTO 170
110 X=A : Y=B
120 FOR K=1 TO 50
130 U=X*X : V=Y*Y : W=2*X*Y
140 X=U-V+A : Y=W+B
150 IF U+V<16 THEN GOTO 170
160 NEXT K
170 L=K MOD 2 : PSET (A,B),L : PSET(A,-B),L
2. **Viewing the Mandelbrot set using Winfract fractal generator.**

Students will create the default Mandelbrot Set, similar to the Figure 10-1, using Winfract Fractal generator.

![Mandelbrot Set](image)

*Figure 10-1*

Students will be encouraged to zoom into and explore different parts of the Mandelbrot Set, especially along the coastal area of the Mandelbrot Set.
Colminating:

Brief review on creating the Mandelbrot set and using Winfract fractal generator.
CHAPTER 5

Summary of Findings, Conclusion and Recommendations

Introduction

This chapter summarizes the content of the mini introduction to fractal geometry course constructed by the researcher. Conclusions on the success of the mini introduction to fractal course for community college students are discussed. The researcher concludes the chapter with recommendations concerning the construction of the development of a regular sixteen week course for community college students.

Summary of Findings

A five week introduction to fractal geometry course lesson plan was constructed by the researcher. The researcher found that topics suitable for community college students in an introduction to fractal course are as follows:

1. Self-similarity
2. Koch Curve
3. Sierpinski's Triangle
4. Functions
5. Iterations
6. Complex Numbers
7. Orbits
8. Classification of Points
9. The Mandelbrot Set

Conclusion

Based on this construction of the lesson plans, an Introduction to Fractal Course for community college students can be developed.

Recommendations

Fractal geometry is a rapidly growing mathematical field which is being applied to new technologies. The mini introduction to fractals course for community college students which was developed by the researcher could be taught to students who are interested in fractals and have the time for a five week course. The success of the course could then be evaluated. If it is successful, it would be worth developing into a regular sixteen week semester course for community college students.
BIBLIOGRAPHY


