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The development of a unit on fractal geometry appropriate for the high school student

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THE DEVELOPMENT OF A UNIT ON FRACTAL GEOMETRY APPROPRIATE FOR THE HIGH SCHOOL STUDENT

by
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ABSTRACT

Ira A. Fine, The Development of a Unit on Fractal Geometry Appropriate for the High School Student, 1995, J. Sooy, Mathematics Education.

The purpose of this study was to develop a three week fractal geometry unit plan and analyze its success at the high school level. The researcher evaluated the research and literature related to fractal geometry and fractal geometry education suitable for the high school student. A fractal geometry unit plan was constructed. Topics covered in the unit plan included introductory concepts, the Koch curve, Sierpinski's triangle, coastline length, the chaos game, complex numbers, the Mandelbrot set and computer generated fractal images.

The unit plan was taught to three tenth grade honors geometry classes at Washington Township High School in Southern New Jersey. Each class consisted of about twentyfive students. The students were given six short quizzes and a final test consisting of twentyfive open ended questions to measure the success of the unit plan. All quiz and test means for the three classes were eightyfour or above. Based on the analysis of the quiz and test data the researcher concluded that the fractal geometry unit plan was successful.

MINI-ABSTRACT

Ira A. Fine, The Development of a Unit on Fractal Geometry Appropriate for the High School Student, 1995, J. Sooy, Mathematics Education.

The purpose of this study was to construct a fractal unit plan and analyze its success at the high school level. A fifteen day unit plan was developed and taught to three high school level geometry classes. This study concluded that the unit plan was successful.

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CHAPTER 1

Introduction to the Study

Introduction

This chapter discusses fractal geometry and the problem of creating a fractal geometry unit plan. The chapter includes background, the statement of the problem, the significance of the problem, the limitations of the study, the definition of terms, and procedures for implementing the study.

Background

Fractal geometry is a new branch of mathematics. Although some of the underlying concepts were discovered by Georg Cantor in the later part of the nineteenth century and Gaston Julia in the early part of the twentieth century, fractal geometry has existed as a subject in its own right for approximately twenty years (Peitgen, Jurgens, & Saupe, 1992) . It became a formal subject largely due to the work of Benoit Mandelbrot in the nineteen seventies (Gleick, 1987) . While working at IBM Research Laboratories he discovered many of the concepts of the subject and invented much of its vocabulary. The use of computers with fractal geometry has added to the development of the subject within the last ten years.

Fractal Geometry has many artistic and practical applications. Computer programs can be used to produce an endless array of colorful

abstract designs based on fractal graphs (Gleick, 1987) . Edward Lorenz has shown how fractal geometry can be used in the prediction of weather patterns (Peitgen et al., 1992) . Many plants, animals and geological formations can best be represented by using fractal models (Peterson, 1990) . "The 19th-century mathematicians may have been lacking in imagination but nature was not " (Mandelbrot, 1983, p. 3). Medical research is beginning to discover how some of the concepts of fractal geometry can be used in conjunction with computers to monitor human heartbeat rates (Goldberger, Rigney, & West, 1990) .

The field of education usually lags several years behind the most recent advances and discoveries in a subject. This is especially true in the field of fractal geometry due to the newness of the subject. For the mathematics teacher who is interested in introducing his or her students to fractal geometry there are very few resources available.

Statement of the Problem

The purpose of this study is to construct and analyze the success of an educational fractal geometry unit plan at the high school level.

Significance of the Problem

The researcher has recently examined several high school algebra and geometry text books. The researcher has found that there is very little material available on fractal education in high school algebra and geometry text books. Almost all of the literature and resources about

fractal geometry address the subject on a level beyond the scope of the high school student and is not of an educational nature.

Fractal geometry is a new branch of mathematics. Many of the concepts and applications of fractal geometry can be taught on a level suitable for the high school student. Therefore there exists a need for a complete fractal geometry unit plan as a resource for the high school mathematics teacher.

Limitations of the Study

There are three major sources of information for this study.

A. Educational and scientific journals with articles pertaining to fractals or fractal education.

B. Books about fractals or fractal education.

C. Information about fractal geometry available on the internet. This includes standard text information and computer programs for generating fractal images.

The prerequisite courses for the fractal geometry unit plan are Algebra I and Geometry (or currently taking Geometry). The unit plan is limited to three weeks of instruction. The unit plan will be piloted with three academically advanced sophomore geometry classes at Washington Township High School. Classes are forty five minutes long and meet five times a week. Computer workstations with BASIC on-line are needed for some of the lessons. Washington Township High School

has a student population of about two thousand five hundred students. The high school has a graduation rate of almost one hundred percent. After graduation approximately forty percent of the students attend a four year college and approximately forty percent of the students attend a two year junior college. Washington Township is a middle and upper middle class suburban community located in Southern New Jersey. The township has experienced tremendous growth in the last fifteen years and presently has a population of about forty five thousand.

Definition of Terms

BASIC - the programming language used in this study which is capable of generating fractal images

fractal - a model in which a part is similar to the whole and this process can be repeated indefinitely with smaller and smaller sections (Peterson, 1990)

fractal geometry - the study of fractals

internet - a global web of computer networks accessible from personal computers

similar - models or drawings which are the same shape but not necessarily the same size.

Procedures

The first phase is to read and analyze the literature and research relating to fractal geometry. Most of the literature and research is in the

form of standard text and some is in the form of computer programs. The second phase of the research is to form a three week fractal geometry unit plan. The third phase is to field test the unit plan during the month of February 1995. The sample in this study is three academically advanced sophomore geometry classes at Washington Township High School. Each of the classes in the study has about twenty five students. The author is the instructor for all classes participating in the study. The last phase of the research is to analyze the success of the unit plan. Success is measured by student performance on tests and quizzes.

CHAPTER 2

Review of Related Research and Literature

Introduction

The primary purpose of this thesis is to construct an educational fractal geometry unit plan suitable for the academically tracked high school student. There is very little research and literature related to fractal geometry education on the high school level. Most of the research and literature concerning fractal geometry does not address the educational aspect of the subject and is presented on a mathematical level beyond the scope of the high school student. There are however several books and articles about fractal geometry which contain relevant information suitable for the high school student. Most of these books and articles are not structured in the form of fractal lessons but are strictly a discussion and explanation of various fractal geometry topics. These sources are discussed in more detail in the related literature section of this chapter.

The internet is another source of information on fractal geometry and computer programs for generating fractal images. These electronically transmitted messages and programs will also be elaborated on in the related literature section of this chapter.

A second purpose of this thesis is to analyze the success of the unit plan in the classroom setting. There is very little research concerning the success that educators have achieved in teaching fractal geometry on the introductory level. Mathematics Teacher has published a few articles

about the teaching of specific fractal geometry topics and to what degree the lessons have met with success. These articles and other sources addressing the educational aspect of fractal geometry will be elaborated in the related research section of this chapter.

Review of Related Research

The Mathematics Teacher is a monthly journal devoted to educational aspects of mathematics. There are four articles from this journal that the researcher finds relevant to this study.

In a 1990 article, Kern and Mauk present a method of introductory fractal instruction. They suggest that the initial concept of fractals be presented in the context of natural phenomenon such as a fern, a twig and a coastline. The authors write that the students should be introduced to the Koch snowflake as an example of a formal fractal shape. Kern and Mauk discuss the use of the computer language Logo as a means of exploring the Koch snowflake with the students. The authors explain how to write an algorithm for generating the Koch snowflake using Logo and then suggest that the students run the program several times using different starting parameters. Students are encouraged to write other programs for generating different fractal curves. It is suggested that the concept of fractal dimension be introduced after the fractal designs have been generated and discussed. The authors explain how fractal dimension differs from classical Euclidean dimension. The formula for calculating fractal dimension is presented and the fractal dimension of

various fractal shapes is computed. The article concludes by mentioning why fractal geometry should be taught on the secondary school level. The authors write that fractal education generates mathematical interest and excitement. They mention that the exploration of fractals can lead the student to the study of other mathematical topics.

In a 1990 article, Barton writes that a method of producing fractal images called the chaos game generated a tremendous amount of interest among his students. The author explained to his students the steps involved in playing the chaos game. Barton suggests having students play the chaos game to determine a few dozen points to make sure that they understand the process involved. Generating a fractal image by playing the chaos game and using pencil and paper is a very tedious procedure and thousands of points would have to be generated in order to produce a fractal image. Students who were familiar with the computer language called Basic were encouraged to write programs to generate fractal images using the chaos game. According to Barton, his students reacted with surprise and interest to the visual images produced by their computer programs. A Basic program for generating fractal images using the chaos game is presented in the article. The author suggests encouraging students to produce other fractal designs by varying some of the parameters in the program. Barton then showed his students how to produce computer images of a fern using fractal generating methods. The author writes that experimenting with the chaos game and computer programs helps students develop an

understanding of randomness and fractal self-similarity.

In a 1991 article, Camp advocates presenting the Koch snowflake as vehicle for introducing fractal geometry. Camp mentions that the procedure for generating the Koch snowflake is an excellent way to introduce the students to the concept of iteration. The author says that the iterative process is used to generate many fractal images and that the students should investigate some of these images on their own. Extending the Koch snowflake from two to three dimensions is very interesting and produces a lot of excitement among students according to Camp. It is suggested that the lessons be taught to a pre-calculus level class. The algorithm for generating the Koch snowflake is explained. The formulas for calculating side length, number of sides, perimeter and area are developed and discussed. The author then suggests that the students be taught the three-dimensional Koch snowflake. Camp explains the algorithm and formulas involved. The best way to demonstrate the three-dimensional Koch snowflake is to construct a model using paper. The author suggests that students be allowed to construct their own three dimensional models and explains the procedure for construction. It is recommended that the students be allowed to discover many of the properties of the three-dimensional Koch snowflake while they are constructing it. Camp suggests allowing students to write their own programs for generating the two-dimensional Koch snowflake and also presents a program written in Pascal if the instructor would rather demonstrate a computer program to the class.

In a 1993 article, Coes says that self-similarity is one of the central ideas of fractal geometry. He say that if the instructor succeeds in teaching students the notions of self-similarity, they will then have a deeper understanding of the more conventional concept of Euclidean similarity. The author suggests using square tiles and interlocking cubes as a method of teaching students on all levels the concepts of self-similarity. First the students should explore the traditional concept of similarity by constructing similar models. This should be done in two and three-dimensional. Coes then describes how to arrange the square tiles so that they begin to form a fractal pattern. The ideas of self-similarity should be discussed with the students as it relates to the models they construct. The author writes that the notion of self-similarity becomes more obvious as the students construct more complex models using the square tiles. After the students have explored and discussed the concept of self-similarity using the square tiles, three-dimensional models exhibiting self-similarity should now be constructed using the solid blocks. In the article Coes includes pictures of how the squares and blocks can be arranged to exhibit self-similarity. The author then suggests introducing the student to the concept of fractal dimension. The method of computing fractal dimension is explained. The students should then calculate the fractal dimensions for the models which they have created. Coes also emphasizes that the students should be made aware that the models which they create are not true fractals because of the fact that the models

are finite. The self-similarity in a truly fractal object can be found an infinite number of times.

Vojack (1989) constructed a packet of fractal geometry lessons intended to be included in a high school pre-calculus class. Vojack's packet is divided into three lessons. The first lesson is an introduction to complex numbers. The second lesson explains complex functions and the concept of iteration. The third lesson concerns itself with Julia sets. All of the lessons include objectives, definitions, examples, exercises and teaching suggestions. A few additional worksheets are also included in the packet. The author does not discuss or mention what degree of success the packet has had as a guide in introductory fractal education.

Peitgen et al. (1991) published a two volume set of fractal geometry lessons for the high school level student. The lessons show how fractals are related to many different branches of mathematics. The authors frequently emphasize the dramatic visual images which fractal geometry can produce. The lessons focus on having the student create the fractal design or pattern which is being explored. Blackline masters are provided as a foundation for student constructions and step by step instructions are included. Each section concludes with several questions about the lesson. Some of the questions require specific mathematical calculation and some are more general in nature, requiring the student to reflect upon the fractal nature of the construction. The end of each volume has a complete answer guide to the questions in each lesson. Volume one is an introduction to fractal geometry and consists of thirty

lessons which are presented in a style which emphasizes the visual interpretation of concepts. Topics include self-similarity, Sierpinski's triangle, Pascal's triangle, fractal trees, the chaos game, and fractal dimension. Volume two extends the exploration of fractals on a more mathematical level. Iteration of specific functions is explored from an algebraic and geometric point of view. The concept of chaos is discussed. Many mathematical models of chaos are presented and graphically represented. Complex numbers are introduced and the Mandelbrot and Julia sets are introduced. Many activities and questions are included which help the student understand the properties of the two sets. The volume concludes with activities about generating the Mandelbrot set on a graphing calculator.

Review of Related Literature

The Fractal Geometry of Nature is a book of significant importance to this study because its author, Benoit Mandelbrot, is the mathematician whom is generally given credit for inventing the field of fractal geometry. A large portion of this book is understandable with only a knowledge of high school algebra. Mandelbrot (1983) discusses the concept that the world we live in is not Euclidean in the sense that it is not composed of straight lines and circles. Instead nature is composed of irregular and random variations in its shape and form. The length of a coastline, the profile of a mountain, the branching of a tree, and the

outline of a cloud are all examples of this phenomenon and are discussed by Mandelbrot (1983).

The Koch curve and the Sierpinski triangle are examples of the concept of iteration. These models are included in the researcher's unit plan and are analyzed by Mandelbrot (1983). Determining the length of a coastline is another introductory fractal problem which is included in the researcher's unit plan and is discussed by Mandelbrot (1983). The researcher includes several lessons on the development of the Mandelbrot set which is named after Benoit Mandelbrot who discovered it. Mandelbrot (1983) explains how the Mandelbrot set is created and its importance to fractal geometry.

Peitgen, Jurgens and Saupe (1992) present a very thorough treatment of most of the topics which are normally covered in an introduction to fractal geometry. Iteration, self-similarity, the Koch curve, the Sierpinski triangle, and the chaos game are topics covered in the researcher's unit plan and are also discussed in detail in the book. Peitgen et al. also include some biographical information about Georg Cantor, Gaston Julia and Waclaw Sierpinski. These are mathematicians whose work contributed to the development of what would become fractal geometry.

The majority of the topics in this work are presented on a mathematical level beyond the scope of the unit plan developed in this thesis. Trigonometry, logarithms and matrices are frequently employed by

the authors. The researcher's unit plan is intended for students who have not yet encountered these topics in their mathematical training.

Barnsley (1993) and Gulick (1992) have written text books which are very complete in their treatment of fractal geometry. These works are presented in the structure of definition, theorem and proof of theorem. The level of mathematical rigor developed by the authors is not suitable for an introductory fractal unit plan intended for the high school student. These text books are suitable for use in a college level course on fractal geometry.

Gleick (1987) and Briggs (1992) are authors of books about a wide range of fractal geometry topics. Gleick and Briggs have written their books on a level which requires no specific knowledge of mathematics. Almost all the ideas developed in these works are presented on a level which does not revert to mathematical equations and formulas. Both of these volumes are an introductory examination of fractal geometry's history, applications and visual manifestations and can be understood by the layman with almost no mathematical training. Gleick goes into great depth developing the notions of strange attractors and chaos. Briggs concentrates more on developing the visual aspects of fractal geometry and includes a large selection of high quality color photograghs depicting fractal images which have been computer generated.

Peterson (1990) and Engel (1989) have published books which are not devoted to the subject of fractal geometry, but do include a chapter which gives the reader a brief introduction into many of the ideas of fractal

geometry. Peterson explains, on a non mathematical level, the Koch snowflake and the Sierpinski triangle. The fractal nature of the world we live in is explained by using mountains, trees and ferns as examples. Engel's chapter also includes a discussion of the Koch snowflake and the Sierpinski triangle and includes a brief explanation of fractal dimension and the Mandelbrot set. Engel's treatment of fractal dimension and the Mandelbrot set does require the reader to have an understanding of algebra and logarithms.

Many articles about fractal geometry have been published in journals and magazines. There are two articles which appeared in Scientific American and are of value to this thesis. Jurgens, Peitgen and Saupe (1990) discuss how fractal geometry can be used to construct models of irregular shaped object in nature. The relationship between the Mandelbrot and Julia sets is explained by the authors. The authors also explain how interesting images can be generated by zooming in on sections of the Mandelbrot set. Dewdney (1990) explains in specific mathematical terms how to define the Mandelbrot and Julia sets. A knowledge of complex arithmetic is needed. General algorithms for generating the Mandelbrot and Julia sets are developed by the author. The relationship between the two sets is explored.

Many articles have been published about the use of fractal geometry with regards to data compression. Data compression, as it applies in this context, refers to the process of reducing the number of bytes required by a computer to reproduce an image. Waters (1989),

Gibbs (1993) and Proise (1994) have all published articles which discuss the process of fractal data compression. The articles explain the basic principle of fractal data compression. A fractal formula is used to recreate a section of the image instead of recording the section pixel by pixel. The authors use specific examples to show how this is accomplished.

Goldberger, Rigney and West (1990) explain how some physiological phenomenon have fractal properties. The authors discuss how the branching of the blood vessels in the human body is of a fractal nature because the vessels repeated divide into smaller blood vessels. The rhythms of the heart are also discussed in fractal terms. The article discusses that if heart rates are looked at as a fractal pattern, this could potentially have benefits for person suffering from various forms of heart disease. The authors mention that medical researchers are just beginning to appreciate the potential value of viewing physiological anatomy and functions in a fractal context.

The Internet is a source of information about fractal geometry. The Internet has a large database of what are referred to as FAQs, which is an abbreviation for frequently asked questions. The FAQ on fractal geometry is a collection of twenty-seven questions and answers relating to fractal geometry (Sherriff, 1994). The FAQ includes such topics as fractal dimension, the Mandelbrot set, the Julia set, complex arithmetic, chaos, strange attractors, and a general reference list of material available about fractal geometry.

The researcher belongs to a fractal internet mailing list. This is group of several hundred persons with an interest in fractal geometry who post questions, answers and comments on a wide range of fractal topics via the internet e-mail system. On the average there are about five messages posted daily. Many of the messages are on a mathematical level beyond the scope of the researcher's thesis. Some of the messages do discuss topics which are of general interest and are of value to the researcher's thesis.

There is a large selection of fractal images available to be downloaded from the internet. These images are computer generated and include an enormous selection of pictures of the Mandelbrot set. The images differ in shape and color because of the parameters which are used to generate them.

The researcher has downloaded from the internet two fractal generating programs. Winfract is a complete fractal generating program (The Stone Soup Group, 1993). Many different fractal generating algorithms can be used and there is a wide selection of parameters from which to choose. Winfract can zoom in on a any section of the fractal image which is selected. Many color schemes can be employed. Winfract requires about three minutes to generate a fractal image.

The Mandelbrot Set is another fractal generating program (Crew, 1993). The Mandelbrot Set has all the features of Winfract except it can only use the Mandelbrot set algorithm. The Mandelbrot Set program can

generate the completed image in about ten seconds. The program also has the ability to animate the image.

Fractals: An Animated Discussion is a one hour video which introduces many of the concepts of fractal geometry (Peitgen, Jurgens, Saupe & Zahlen). The film includes an interview with Benoit Mandelbrot. Mandelbrot discusses the basic meaning of what a fractal is and its relationship to living and nonliving objects in nature. The film explains many types of fractals and illustrates how they can be generated on a computer.

CHAPTER 3

Procedures

Introduction

The purpose of this chapter is to explain the procedures the researcher used to develop a three week unit plan in fractal geometry and the procedures used to evaluate the success of the unit plan. The topics discussed in this chapter include procedures used in the areas of selection of related material , selection of topics included in the unit plan, development of individual lessons, construction of related computer programs, and methods used to evaluate the success of the unit plan. The instructional setting and population used to field test the unit plan is also profiled in this chapter.

Selection Related Material

The researcher selected material for this study from several sources. All material that could be located which the researcher considered suitable for developing a fractal unit plan for the high school level student was reviewed. This included articles from magazines and journals, books related to fractal geometry and computer programs for generating fractal images. The researcher's primary sources for locating this material were Savitz Library at Rowan College of New Jersey, Washington Township High School Library in Sewell, New Jersey, and

material downloaded from the Internet using Veronica and Archie searches.

Selection of Fractal Topics

After the researcher had evaluated the related material, the decision had to be made as to what fractal topics should be included in the unit plan. Within the limits of a three week time frame there are many aspects of fractal geometry which can not be included. The researcher used seven criteria to determine whether or not a topic should be included in the unit plan:

1. Can the topic be presented on a level which can be understood by academically tracked high school students who have completed at least one year of high school algebra?
2. Can the topic be exhibited by student construction using tools such as pencil, paper, ruler and protractor?
3. Can the topic be exhibited by a computer generated image?
4. Does the topic help introduce or reinforce the notion of iteration?
5. Is an understanding of the topic necessary for the students to comprehend the Mandelbrot set?
6. Can the topic adequately be presented in no more than three class periods?
7. Is the topic generally included in an introductory discourse of fractal geometry?

The researcher included topics in which the answer was yes to at least five of the seven questions.

Development of Individual Lesson Plans

The researcher structured each lesson so that it could be presented in fortyfive minutes. Each lesson plan included new definitions, concepts and examples related to the topic under consideration. The lesson plans were also written to include step by step procedures for students to follow in order to construct the various fractal patterns and designs. The scope and sequence of the lesson plans was designed so that each lesson would be a logical outgrowth from the previous lesson. Topics covered ranged in duration from one to three lesson plans. The lesson plans were developed so as to include a short four or five question quiz on the day following the completion of a topic. Homework was included in most lesson plans. Some homework assignments took the form of completing a project begun by the students as classwork. Other homework assignments involved questions involving the mechanics of a particular construct or questions designed to develop understanding of fractal concepts. A central theme throughout all the lesson plans is the cultivation of the notion of iteration.

Two lesson plans were developed to introduce the students to the arithmetic of complex numbers and graphing complex numbers in the complex plane. Although complex numbers are not a topic in fractal geometry, an understanding of them is a necessary tool for constructing

the Mandelbrot set.

The researcher wrote three computer programs which were designed to exhibit some of the concepts presented in the lessons. These computer programs were incorporated into lessons about computer related fractal activities and will be discussed in more detail in the following section of this chapter.

Construction of Related Computer Programs

The researcher wrote three computer programs to be used in a lessons whose purpose was to introduce students to computer generated fractal images. The three programs were designed to help students gain an understanding of the following topics:

1. The Mandelbrot Set
2. Sierpinski's Triangle
3. Random movement

All the programs were written in Quick Basic because this is a computer language which has good graphic capabilities and is commonly available in high school computer laboratories.

The researcher developed the Mandelbrot set and the Sierpinski triangle programs so as to follow the same method of generation used by the students in the classroom. Another factor in constructing these programs was to keep the algorithms as simple as possible so that their basic structures could be understood without any prior computer programming knowledge.

The random movement program was written by the researcher to show the students how a computer could be used to generate random movement. In this program the researcher starts with a point at the center of the screen. Then the computer is directed to select a random number to correspond to a direction in the XY-plane. This process is repeated continuously to generate a random path. The researcher also added a mirror image feature to this program to make it more interesting from visual perspective.

All of the Quick Basic programs used for the unit plan were constructed by the researcher on a Pentium ninety megahertz IBM compatible computer. The programs were transferred to a floppy disc and then downloaded onto the computer network used by the researcher to field test the programs with the students. The computer laboratory used by the researcher in this study consisted of twentyfour networked workstations. Each workstation consisted of one thirtythree megahertz 486 IBM compatible computer.

Evaluation Procedures

The success of the unit plan was measured by the researcher using two methods. The first method was a series of six quizzes. Each quiz was administered during the first ten minutes of a class period after the conclusion of a topic. Five of the quizzes consisted of four questions. One of the quizzes consisted of five questions. The researcher kept a running total for each student of the number of quiz points accumulated.

Each student's quiz total was multiplied by four to convert the score to a hundred point scale because there was a potential of twenyfive total quiz points.

The second method of evaluation was a test consisting of twentyfive open ended questions. The researcher constructed the test so that each topic included in the unit plan would be represented. The test was administered in one fortyfive minute period at the conclusion of the unit plan. Each student's number of correct answers was multiplied by four to convert the score to a hundred point scale.

The data consisting of the cumulative quiz score and the test score for each student was then analyzed. Analysis consisted of computing the mean, median, mode, and standard deviation for the cumulative quiz and test scores. These results are produced in chapter four of this thesis.

Instructional Setting

The researcher field tested the unit plan on three academically advanced tenth grade geometry classes at Washington Township High School in Southern New Jersey. Washington Township High School has a student population of about two thousand five hundred students. About eighty percent of the students attend college after graduation from the high school. The three classes met Monday through Friday during the first three periods of the school day. The classes had enrollments of twentyfive, twentyfour and twentythree students. The unit plan was field tested during the month of January 1995.

CHAPTER 4

Analysis of Data

Introduction

This chapter describes the various topics in fractal geometry which the researcher chose to include in the unit plan. The actual lesson plans, quizzes, test, answer keys, diagrams, computer programs and computer printouts can be found in the appendixes of this study. The students' performance on the quizzes and test is also analyzed and discussed in this chapter.

Fractal Geometry Topics Included in the Unit Plan

The first topic included in by the researcher is an introduction to the meaning of fractal geometry. The concept of self similarity is introduced and discussed through the use of real world and geometric examples. The definitions of a fractal object and fractal geometry are presented. Some of the applications of fractal geometry are included in this lesson.

The next two topics covered are the Koch snowflake and the Sierpinski triangle. These fractal shapes are an excellent method of introducing students to the iterative process. Methods of constructing a Koch snowflake and a Sierpinski triangle are presented. Students construct and then analyze the geometric properties of the shapes. The fractal nature of the Koch snowflake and the Sierpinski triangle are discussed.

The chaos game follows the construction of the Sierpinski triangle because it is a different method of forming the Sierpinski triangle and also introduces students to the concepts of randomness and chaos. The chaos game is an interesting method of generating the Sierpinski triangle because it can easily be translated into a computer program.

The length of an island's coastline is the next topic included in the unit plan. Various ways of transversing a coastline are discussed and a comparison of their corresponding lengths is analyzed. Students then calculate the perimeter of an island using several different units of measurement. This problem stimulates thought about the fractal nature of real world objects.

Three introductory lessons on complex numbers are then presented. A basic understanding of complex number theory is necessary in order to comprehend the Mandelbrot set. Topics covered are i, complex numbers, complex number arithmetic and graphing complex numbers in the complex plane. It is assumed that the students have had no prior experience with complex numbers.

The presentation of the Mandelbrot set follows the introduction to complex numbers. The definition of the Mandelbrot set is developed and several points in the complex plane are tested to determine whether or not they are elements of the Mandelbrot set. As the number of known points included in the Mandelbrot set increases, the general location of the Mandelbrot set begins to emerge.

The last topic covered in the unit plan is computer generated fractal images. Students are familiarized with computer programs which generate random designs, Sierpinski's triangle and the Mandelbrot set. The final activity takes place in the computer lab where students run the programs and then observe and discuss the computer generated fractal images.

Analysis of Quiz and Test Scores

Seventythree students were given six short quizzes and one final test during the course of the unit plan. The purpose of these instruments was to provide a method of measuring the success of the unit plan. The quiz and test results are presented in the following table.

Table 1

Achievement on quizzes and test

Quiz or test	Mean	Median	Mode	Range	S.D.
Introduction	95.9	100	100	50	10.2
Koch snowflake	96.9	100	100	25	8.3
Sierpinski triangle	97.6	100	100	25	7.4
Coastline/chaos	96.2	100	100	50	10.8
Complex numbers	84.1	100	100	80	21.1
Mandelbrot set	91.8	100	100	50	12.5
Final test	86.1	88	96	40	10.0

The possible scores for each quiz, with the exception of the Mandelbrot quiz, are 0, 25, 50, 75, and 100. The possible scores for the Mandelbrot quiz are 0, 20, 40, 60, 80, and 100. The final test consists of twentyfive free response questions, each worth four points. The mean score for the each of the first four quizzes is about twelve points higher than the mean score for the complex number quiz and about five points higher than the mean score for the Mandelbrot quiz. It is difficult to compare the mean test score to the mean quiz scores because of the differences of the two instruments. The quizzes are short and address only what was taught in the previous lesson. The test however is a much more comprehensive device and covers material presented over a three week period.

CHAPTER 5

Summary of Findings, Conclusions, and Recommendations

Introduction

This chapter summarizes the content of the fractal geometry unit plan constructed by the researcher. A summary of quiz and test results is also presented. Conclusions are discussed about the success in developing and teaching a fractal unit plan suitable for the high school level. The researcher concludes this chapter with recommendations concerning the development of other fractal geometry lesson and unit plans.

Summary of Findings

A three week fractal geometry lesson plan was developed by the researcher. The topics covered in the unit included

1. introductory vocabulary and ideas,
2. the Koch snowflake,
3. Sierpinski's triangle,
4. the chaos game,
5. coastline problems,
6. complex numbers,
7. the Mandelbrot set, and
8. computer programs.

The length of time necessary to teach a topic varied from one to three lessons.

The mean score for each of the first four quizzes was about ninetysix. The mean scores for the complex number and Mandelbrot set quizzes was about eightyfour and ninetytwo. The mean score for the final test was about eightysix.

Conclusions

Based on the lesson plans developed in this study, a fractal geometry unit plan suitable for the high school can be developed.

The researcher field tested the unit plan in Washington Township in Southern New Jersey. At Washington Township High School a grade of seventy represents a successful completion of a quiz, test or course. In this study the researcher accepts this criterion as the measure of success in the teaching of the fractal geometry unit plan. Based on the analysis of the quiz and test results in this study, the fractal geometry unit plan was successfully taught.

Recommendations

Fractal geometry unit plans can vary greatly in length, topics covered and level of mathematical maturity. It would be a very worthwhile undertaking to construct other fractal geometry unit plans which address these differences.

The unit plan in this study is three weeks in length. A unit plan of longer duration could more fully explore the topics covered in this unit plan and also include fractal topics of interest not present in this study. It

would be very stimulating for the student to spend more time in the computer laboratory. Lessons designed to show students how to construct beautiful and colorful fractal images on a computer would be an interesting topic for further study. The fractal nature of clouds, tress and other real world objects would also be on area for further lesson plan development.

Another interesting and worthwhile direction to explore is the construction of fractal geometry unit plans for the middle and elementary school level student. Although some of the mathematics necessary for an understanding of the Mandelbrot set is above the level of this age group, many of the other fractal concepts and designs can be comprehended and appreciated by middle and elementary school students.

Appendix A

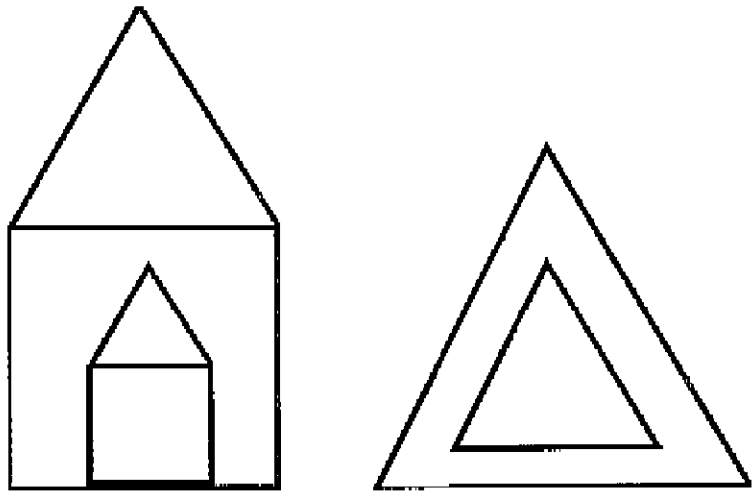
Fractal Geometry Unit Plan

Each of the following lesson plans is designed to cover a time period of about fortyfive minutes. This includes time for presentation of concepts, student activities, discussion, and quizzes. It is suggested that each of the quizzes in appendix B be given at the beginning of the class period following the completion of the topic. Students should have or be supplied with rulers and protractors in order to properly complete the assignments in the first five lessons. All calculations are carried out to four decimal point accuracy.

LESSON ONE: INTRODUCTION

Self Similarity - if a part of an object is exactly or roughly the same shape as the object then the object exhibits self similarity.

Here are drawings of two geometric examples:

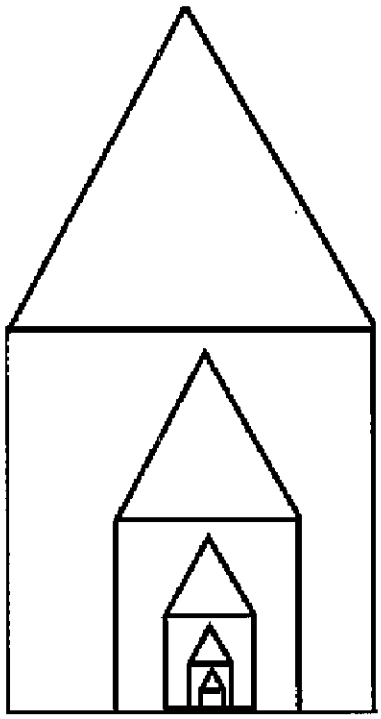


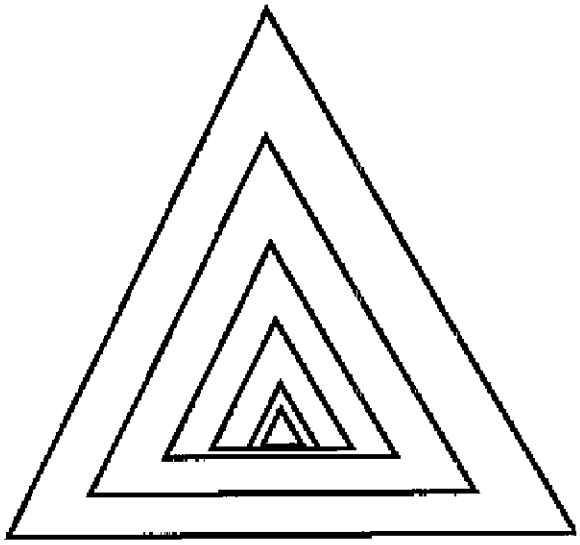
Two real world examples are:

1. a tiled square room
2. a kangaroo with a joey in the pouch

Fractal Object - an object in which self similarity can be found over and over again with smaller and smaller sections of an object.

Here are two geometric fractal drawings. It should be stressed that the drawings are meant to get infinitely small.





Real world examples of fractal shapes are:

1. the branching of a tree
2. blood vessels in the human body
3. a head of cauliflower
4. the reflection in two mirrors facing each other

Fractal Geometry - the mathematical study of fractal objects.

Benoit Mandelbrot - the mathematician who developed many of the ideas and concepts of fractal geometry about twenty years ago.

Uses of fractal geometry:

1. medical research with heart problems
2. weather prediction
3. abstract art through the use computer images
4. data compression for computer file transfers.

Suggested activities after presentation of this lesson are:

1. Have students list other real world examples of self similarity and fractal objects.
2. Students should draw other examples of geometric shapes exhibiting self similarity and shapes having fractal properties.
3. Encourage students to do a research report on one the mentioned uses of fractal geometry.

LESSON TWO: THE KOCH SNOWFLAKE

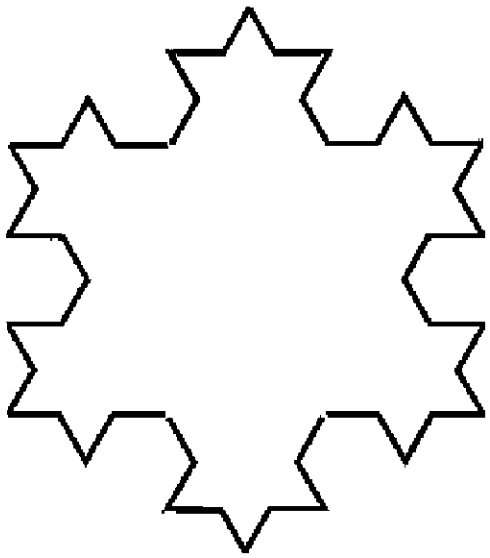
The Koch Snowflake - a very popular fractal shape invented by the Swedish mathematician Helge von Koch in 1904.

Iteration - the process of repeating the same process over and over again on the result of the process.

The Koch snowflake is generated be using the following iterative process.

1. Start with an equilateral triangle.
2. Divide each side into thirds.
3. Construct a new equilateral triangle on the middle third of each side.
4. Erase the middle third of each side.
5. Continue to repeat steps 2, 3 and 4.

Here is the Koch snowflake drawn through two iterations.



Suggested activities after the presentation of this lesson are:

1. Have students construct a Koch snowflake, repeating the iterative process at least three times.
2. Present the students with the following questions:

   How does the number of sides compare to the number of sides of the

   previous level? Answer: it increases four times

   How does the length of each side compare to the length of each side of the previous level? Answer: it decreases by a factor of three

   How does the perimeter of each level compare to the perimeter of the previous level? Answer: it increases by a factor of 4/3
3. Have students construct a chart for the first four levels of the Koch snowflake showing number of sides, length of each side, and perimeter.

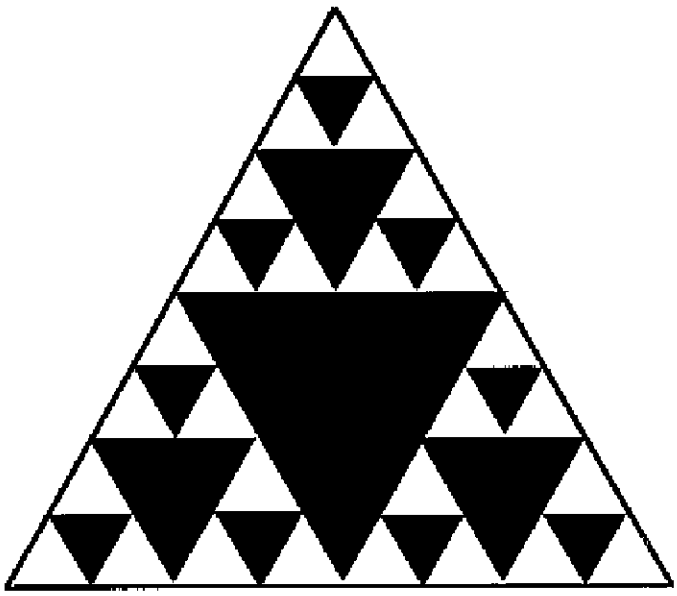
LESSON THREE: SIERPINSKI'S TRIANGLE

Sierpinski's Triangle - this is another very popular fractal design which can be constructed with pencil and paper.

The Sierpinski triangle can be generated using the following iterative process:

1. Start with an equilateral triangle which is not shaded in.
2. Locate the midpoint of each side of every unshaded triangle.
3. Connect the three midpoints of each unshaded triangle.
4. Shade in the new triangle(s) formed.
5. Continue to repeat steps 2, 3, and 4.

Here is a drawing of the Sierpinski triangle with the iterative process repeated three times



Suggested activities after the presentation of this lesson are:

1. Have students construct a Sierpinski triangle, repeating the iterative process four times.
2. Present the students with the following questions:

   How does the number of unshaded triangles compare to that of the previous level? Answer: it increases three times

   How does the area of each unshaded triangle compare to that of the previous level? Answer: it decreases by a factor of four

   How does the total unshaded surface area compare to that of the previous level? Answer: it is 3/4 of the previous area
3. Have students construct a chart for the first four levels of the Sierpinksi triangle showing the number of unshaded triangles, the area of each unshaded triangle, and the total area of the unshaded triangles.

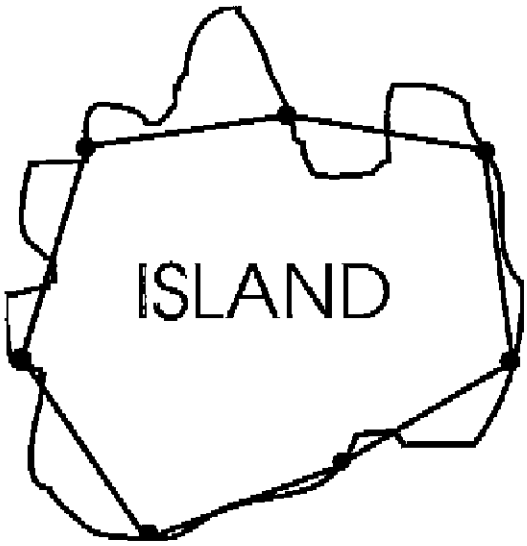
LESSON FOUR: COASTLINE PROBLEMS

A popular fractal problem is to calculate the length of the coastline of an island. In this lesson mythical islands are used. There are several methods which could be used to compute the length of the coastline.

1. An ocean liner could sail around the island.
2. A speed boat could travel around the island hugging the coastline as closely as possible.
3. A person could walk along the coastline.
4. An ant could crawl along the coastline.

The coastline length measured by the ocean liner would be the smallest and that measured by the ant would be the largest. As the unit of measurement decreases, the calculated coastline length increases.

Here is a drawing of a mythical island with the perimeter calculated with a one inch measuring unit.



Suggested activities after the presentation of this lesson are:

1. Draw a mythical island and give each student a copy. Have the students measure the coastline using a three, two and one inch unit of measurement.
2. Have each student draw their own island and again compute its coastline using a three, two and one inch unit of measurement.
3. Have students make statements about what happens to the coastline length as the unit of measurement decreases.

LESSON FIVE - THE CHAOS GAME

Chaos - the concept that very small data changes can dramatically influence the result of a process or experiment. Another notion of chaos is that randomly chosen events can produce an orderly result.

An excellent example of randomness producing order is the chaos game. The chaos game generates the Sierpinski triangle through a process of random selection. In order to play the chaos game the students will need a die or some other method of making a random selection of three choices. Rolling a one or two represents choice A. Rolling a three or four represents choice B. Rolling a five or six represents choice C. The Sierpinski triangle can be generated from an equilateral triangle using the following iterative process:

1. Randomly select a point P in the interior of the triangle.
2. Randomly select one of the vertexes of the triangle.
3. Find the midpoint of the randomly selected vertex and point P.
4. Plot this midpoint and call it P.
5. Go to step 2.

This process will generate points(P) which will never fall in the shaded area of the Sierpinski triangle. If the process is repeated many times the Sierpinski triangle will begin to form. The Sierpinski triangle can easily be generated using the chaos game on a computer. This is one of the programs in lesson eleven.

Suggested activities after the presentation of this lesson are:

1. Have students play the chaos game and generate at least ten points.
2. Have students construct the first two levels of the Sierpinki triangle using the method in lesson three. This should be drawn on top of the ten points generated using the chaos game.
3. Students should notice that the ten points generated do not fall within the shaded regions generated using the lesson three method.
4. Students should experiment playing the chaos game with other shaped triangles. The Sierpinski triangle can successfully be made with any type of triangle.

LESSON SIX: INTRODUCTION TO COMPLEX NUMBERS

$x^2 = 9$, $x = 3$ or -3

$x^2 = 1$, $x = 1$ or -1

$x^2 = -1$ has no real solution because any positive number squared equals a positive number and any negative number squared also equals a positive number.

We define i to equal the square root of -1. That is, $i^2 = -1$.

i is called an imaginary number.

A complex number is defined as any number which can be written in the form a + bi, where a and b are real numbers.

Examples of complex numbers are:

$6 + 2i$

$-3 + 5i$

$2 - 7i$ or $2 + -7i$

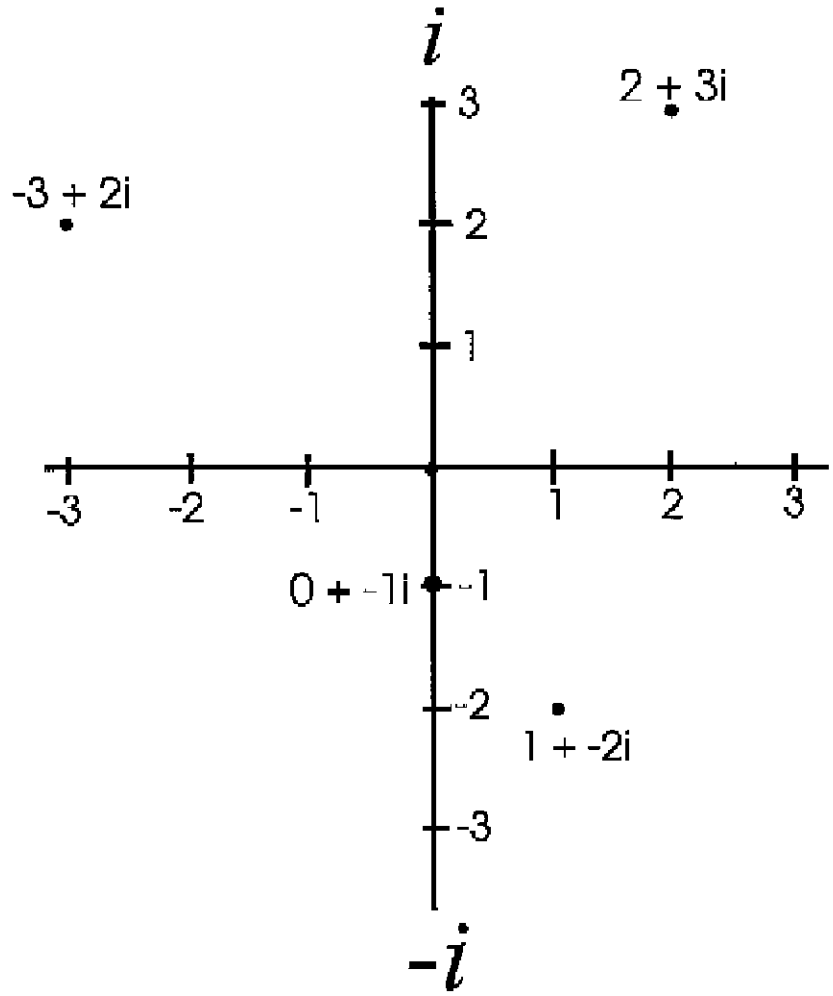
$-19.2 - 3.6i$ or $-19.2 + -3.6i$

Complex numbers can be graphed in what is called the complex plane.

The x-axis is the real part of the complex number.

The y-axis is the imaginary part of the complex number.

Therefore any complex number, a + bi would be graphed as (a,b) in the complex plane as displayed in the following graph.



Complex numbers can be added and subtracted using the following rules:

$(a + bi) + (c + di) = (a + c) + (b + d)i$

and

$(a + bi) - (c + di) = (a - c) + (b - d)i$

Examples:

$(2 + 3i) + (5 + 6i) = (2 + 5) + (3 + 6)i = 7 + 9i$

$(-8 + 2i) + (3 + 3.4i) = -5 + 5.4i$

$(4 + 3i) - (3 + 7i) = (4 - 3) + (3 - 7)i = 1 + -4i$

$(2 + 8i) - (5 - 6i) = (2 - 5) + (8 - -6)i = -3 + 14i$

Suggested activities after the presentation of this lesson are:

1. Have students graph the following complex numbers:

   3 + 6i, -2 + 4i, -3 + -5.5i, 7 - i, -6 - 9i, -7, and 3i

2. Have students solve the following addition and subtraction problems

   1. (4 - 3i) + (6 + 2i) answer: 10 + -i or 10 - i
   2. (3 + 2.2i) + (4- 7i) answer: 7 + -4.8i or 7 - 4.8i
   3. (7 - 6i) + (4 - 2i) answer: 11 + -8i or 11 - 8i
   4. (2 + 3i) + (7 + 2i) answer: 9 + 5i
   5. (7.2 + 20i) + (-3 - 5i) answer: 4.2 + 15i
   6. (2 + i) + (8 -3i) answer: 10 + -2i or 10 - 2i
   7. (4 + 3i) - (8 + 2i) answer: -4 + i
   8. (2 - 4i) - (6 - 3i) answer: -4 + -i or -4 - i
   9. (7 + 2i) - (3 + 5i) answer: 4 + -3i or 4 - 3i
   10. (3 + 3i) - (8 + 2.5i) answer: -5 + .5i

LESSON SEVEN: MULTIPLYING COMPLEX NUMBERS

Recall from algebra I that $(a + b)(c + d) = ac + ad + bc + bd$

Therefore $(a + bi)(c + di) = ac + adi + bci + bdi^2 = ac + (ad + bc)i + bd(-1)$

$= ac + (ad + bc)i - bd = (ac - bd) + (ad + bc)i$

Examples:

1. $(2 + 3i)(4 + 5i) = 8 + 10i + 12i + 15i^2 = 8 + 22i + 15(-1) = -7 + 22i$
2. $(6 - 2i)^2 = (6 - 2i)(6 - 2i) = 36 - 12i - 12i + 4i^2 = 36 - 24i + 4(-1) = 32 - 24i$
3. $(3 + 2i)^2 + (7 - 3i) = 9 + 6i + 6i + 4i^2 + 7 - 3i = 9 + 6i + 6i + 4(-1) + 7 - 3i$
   $= 12 + 9i$

Here are three problems to do with the students as class work:

1. $(3 + 6i)(5 + 2i)$ answer: $3 + 36i$
2. $(5 - 4i)^2$ answer: $41 - 40i$
3. $(2 + i)^2 + (7 - 2i)$ answer: $10 + 2i$

Suggested activities after the presentation of this lesson are:

A. Have students solve the following multiplication problems:

   1. $(2 + 3i)(5 + 7i)$ answer: $-11 + 29i$
   2. $(3 - i)(4 + 9i)$ answer: $21 + 5i$
   3. $(4 - 5i)(8 - 6i)$ answer: $2 - 64i$
   4. $(7 + 2i)^2$ answer: $45 + 28i$
   5. $(.4 - 3i)^2$ answer: $-8.84 - 2.4i$
   6. $(3 + 5i)^2 + (4 - 9i)$ answer: $-12 + 21i$

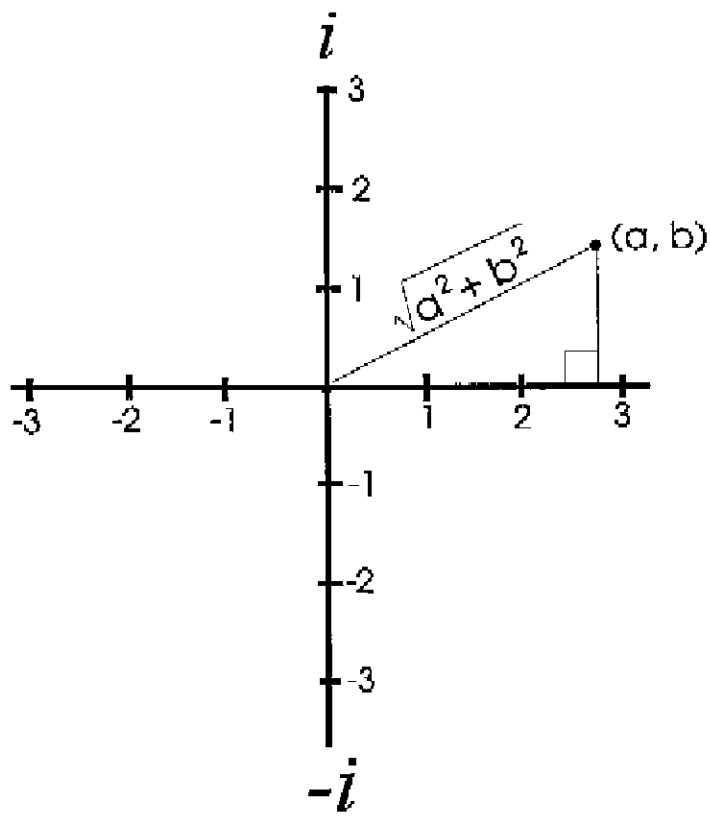
B. Have students write and solve six complex number multiplications

LESSON EIGHT: COMPLEX NUMBER MAGNITUDE

Magnitude of a complex number - the distance a complex number is from the origin.

Recall that in a right triangle the Pythagorean Theorem states that $a^2 + b^2 = c^2$, where a and b are the lengths of the legs of the triangle and c is the length of the hypotenuse.

Therefore the magnitude of any number complex number a + bi is equal to $\sqrt{a^2 + b^2}$ as shown in the following graph.



Examples:

Find the magnitude of 3 + 4i, $\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

Find the magnitude of 2 + 0i, $\sqrt{2^2 + 0^2} = \sqrt{4} = 2$

Find the magnitude of .5 - .6i, $\sqrt{.5^2 + .6^2} = \sqrt{.25 + .36} = \sqrt{.61} = .7810$

Have students compute the magnitude of the following complex numbers as class work.

1. .4 - 2i answer: 2.0396
2. 12 + 5i answer:13
3. .7 + .3i answer: .7616
4. 32 + 0i answer: 5.6569

Suggested activities after the presentation of this lesson are:

A. Have students compute the magnitude of the following numbers:

   1. 3 - 4i answer: 5
   2. .1 + .1i answer: .1414
   3. 4 + .2i answer: 4.0050
   4. .61 + .3i answer: .6798
   5. .27 - 1.2i answer: 1.23

B. Have students write and compute the magnitude of five complex numbers.

LESSON NINE: THE MANDELBROT SET

The Mandelbrot set is constructed by testing points in the complex plane to see whether or not they are part of the set. The test consists of the following iterative process.

1. Start with 0 + 0i.
2. Add the complex number being tested.
3. Compute the magnitude of this sum.
4. Square the sum.
5. Go to step 2.

If the magnitude of the sum gets greater and greater as the loop is repeated over and over again, then the complex number being tested is not in the Mandelbrot set.

If the magnitude of the sum ever gets greater than two, this means that the magnitude of the sum will get greater and greater and therefore the complex number being tested is not in the Mandelbrot set.

There is no definitive number of times the iterative process should be repeated to guarantee that a test number is in the Mandelbrot set. The more times the iterative loop is repeated without the magnitude of the sum becoming greater than two, the more likely it is that the test number is in the Mandelbrot set.

In the following exercises the test number will be checked up to three times through the loop to see if its magnitude is greater than two. The step numbers in the examples refer to the step numbers in the iterative loop explained on the previous page.

Example 1: Test number = 1 + 1i

1. 0 + 0i
2. 0 + 0i + 1 + 1i = 1 + 1i
3. $\sqrt{1^2 + 1^2} = \sqrt{1 + 1} = \sqrt{2} = 1.4141$ (1.4141 < 2)
4. $(1 + 1i)^2 = 1 + 2i - 1 = 2i$
2. 2i + 1 + 1i = 1 + 3i
3. $\sqrt{1^2 + 3^2} = \sqrt{10} = 3.6122$ (3.1622 > 2)

(3.1622 > 2) tells us that 1 + 1i will produce greater and greater magnitudes as the loop is repeated. Therefore 1 + 1i is not in the Mandelbrot set.

Example 2: Test number = 0 + .5 i

1. 0 + 0i
2. 0 + 0i + 0 + .5i = 0 + .5i
3. $\sqrt{0^2 + .5^2} = \sqrt{.5^2} = .5$ (.5 < 2)
4. $(.5i)^2 = .25i^2 = -.25$
2. -.25 + 0 + .5i = -.25 + .5i
3. $\sqrt{.25^2 + .5^2} = \sqrt{.0625 + .25} = \sqrt{.3125} = .5590$ (.5590 < 2)
4. $(-.25 + .5i)^2 = -.1875 - .25i$

2. $-.1875 - .25i \quad + \quad 0 + .5i \quad = \quad -.1875 + .25i$

3. $\sqrt{.1875^2 + .25^2} = \sqrt{.0352 + .0625} = \sqrt{.0977} = .3126 \qquad (.3126<2)$

After three times through the loop the test number 0 + .5i has not produced a magnitude greater than two. Therefore, based on three iterations, 0+ .5i is in the Mandelbrot set.

Suggested activity after presentation of this lesson:

Have students calculate if each of the following numbers is in the Mandelbrot set based on a maximum of three iterations. These problems take about five minutes each and should be completed for homework.

1. -2 + 3i answer: not in Mandelbrot set
2. -1 - 1i answer: not in Mandelbrot set
3. 0 + .5i answer: in Mandelbrot set
4. -1 + 0i answer: in Mandelbrot set
5. -1 + .5i answer: not in Mandelbrot set

LESSON TEN: THE MANDELBROT SET CONTINUED

In order to appreciate the Mandelbrot set it is very important that the iterative process involved is understood. For this reason it is suggested that a second day be spent testing numbers to see if they represent points which are in the Mandelbrot set. Again, a maximum of three iterations should be used.

Have students put the homework problems on the chalkboard and check their work.

Assign the following additional problems as class work.

1. -1 -1i answer: not in Mandelbrot set
2. 1 - 1i answer: not in Mandelbrot set
3. 0 - .5i answer: in Mandelbrot set
4. 3 + 2i answer: not in Mandelbrot set
5. 0 + .1i answer: in Mandelbrot set

LESSON ELEVEN: COMPUTER PROGRAMS

The purpose of this lesson is to introduce the students to fractal images generated by a computer. The four programs I have written for this lesson are meant to be executed on a IBM compatible computer with Microsoft Quick Basic installed on the system. It is most advantageous for each student to have his or her own work station. Students can however pair up or the entire lesson could be presented by the instructor using a single computer. The programs can be typed individually by the students or the programs can preloaded if the system is networked. If the instructor is not familiar with the workings of the computer lab, the lab supervisor should be contacted for assistance.

The Programs

1. Random Image Generator: This program can be found on page 67 in appendix D. The purpose of this program is to demonstrate how a computer can be used to generate a random image. The program mirror images the display with respect to the Y-axis to make the output visually more interesting. A sample output of this program can be found on page 71 in appendix E. The students can change the color of the image by changing the number 15 in line 63 to any other number 1 through 14.
2. Sierpinski's Triangle: This program can be found on page 68 in appendix D. The purpose of this program is to demonstrate how a

computer can be used to generate the Sierpinksi triangle using the chaos game. The output of this program can be found on page 72 in appendix E. The colors used can be changed in lines 120, 210, and 310 as described in the random image generator program.

3. The Mandelbrot Set: This program can be found on page 69 in appendix D. The purpose of this programs is to demonstrate how a computer can be used to generate the Mandelbrot set. The colors can be changed in lines 60 and 63 as described in the random image generator program. This program uses a maximum iteration number of 25. The output of this program can be found on page 73 in appendix E. The output of this program with the iteration number in line 47 changed to 100 iterations can be found on page 74 in appendix E. This program has to run for a few minutes before the first Mandelbrot set points appear on the screen.
4. Zoomed in Mandelbrot Set in Color: This program can be found on page 70 in appendix D. The output of this program can be found on page 75 in appendix E. The purpose of this program is to demonstrate how a computer can be used to generate a colorful variation of the Mandelbrot set. This is accomplished by zooming in on a small section of the set near its border and by assigning different colors to points based on what iteration level their magnitude became greater than two.

There are two very important properties that should be pointed out after the completion of this lesson.

1. All Mandelbrot set points are within a distance of two units from the origin.
2. The Mandelbrot set is a fractal image because smaller and smaller Mandelbrot sets can be found near its border.

Note: The program to generate the Mandelbrot set using 100 iterations and program four take a few hours to generate. Therefore it is suggested that the instructor start these programs on a few computers early in the period so the students can see the images beginning to develop by the end of the period. The students should then be shown the final output of these programs.

LESSON TWELVE: REVIEW FOR TEST

The purpose of this lesson is to provide a review for the fractal geometry test.

1. Students should review and discuss the meaning of the following words and concepts: self similarity, fractal object, fractal geometry, Benoit Mandelbrot, Koch snowflake, Iteration, Sierpinski triangle, chaos, the chaos game, Mandelbrot set, imaginary and complex numbers.
2. Students should review and discuss the iterative process for constructing the Koch snowflake, the Sierpinski triangle, the chaos game and the Mandelbrot set.
3. Students should review and discuss the properties of the Koch snowflake, the Sierpinski triangle and the Mandelbrot set.
4. Students should review and discuss the method of graphing, adding, subtracting, multiplying and finding magnitudes of complex numbers.
5. Students should review and discuss some of the uses of fractal geometry.
6. Students should review and discuss the methods of calculating the length of the coastline of an island.

LESSON THIRTEEN: FRACTAL GEOMETRY TEST

This test is designed to be given in a class period of fortyfive minutes. The test can be found on page of appendix B.

LESSON FOURTEEN: FRACTAL GEOMETRY FILM

Show the film Fractals: An Animated Discussion by H. O. Peitgen, H. Jurgens, D. Saupe and C. Zahlten. This is an excellent film to use as a concluding activity to this unit plan. The film discusses the meaning of fractal geometry, shows many interesting computer generated fractal images and has an interview with Benoit Mandelbrot. The film is sixty-three minutes long. It should therefore be concluded on the final day of the unit plan.

LESSON FIFTEEN: CONCLUSION OF FILM AND TEST RETURN

1. Show the concluding twenty minutes of Fractals: An Animated Discussion.
2. Pass back and go over test answers.

Appendix B

Quizzes and Test

QUIZ 1 - INTRODUCTION TO FRACTAL GEOMETRY

1. If part of an object is the same shape as the entire object then the object is _____.
2. If smaller and smaller copies of an object can be found within an object then the object is a _____ object.
3. True or False

   An apple is an example of a fractal object.
4. Name the person who developed many of the ideas of fractal geometry.

QUIZ 2 - THE KOCH SNOWFLAKE

1. Name the Swedish mathematician who in 1904 invented the fractal snowflake.
2. If a fractal snowflake has a side length of 6 inches in level 2, what will be a side length in level 3?
3. If more and more levels of the snowflake are created its perimeter

   a) remains constant b) equals 50 units c) approaches 0

   d) gets larger and larger
4. True or False

   The construction of the fractal snowflake begins with an equilateral triangle.

QUIZ 3 - THE SIERPINSKI TRIANGLE

1. True or False

   The Sierpinski triangle is formed by bisecting the three angles of the triangle.
2. With each new level, how many times does the number of unshaded triangles increase?
3. The total unshaded area of each new level is what fraction of the unshaded area of the previous level?
4. Who invented the Sierpinski triangle?

QUIZ 4 - CHAOS GAME AND COASTLINE PERIMETER

1. True or False

   As the unit of measurement decreases used to measure a coastline, the computed length of the coastline increases.
2. The idea that very small changes in data can greatly effect the outcome is an example of _____.
3. If the chaos game is played long enough _____ will begin to appear.
4. How many points are you randomly choosing from in the chaos game?

QUIZ 5 - COMPLEX NUMBERS

1. $(3 + 2i)(4 - 3i) =$
2. $(2 + 6.3i) + (5 - 4.7i) =$
3. $(9 - 3i) - (7 - 4.9i) =$
4. $(1.3 + 2i)^2 =$
5. Find the magnitude of $1.8 + .3i$ (Round answer to nearest hundredth)

QUIZ 6 - THE MANDELBROT SET

1. The Mandelbrot set was discovered about _____ years ago.
2. If the magnitude of the iterated test point ever gets larger than _____ it means that the test point is not in the Mandelbrot set.
3. True or False

   The entire Mandelbrot set can easily be generated with ruler, pencil and paper in a few minutes.
4. If the point $0 + i$ is iterated to see if it is in the Mandelbrot set, what is the length of the third magnitude?

FRACTAL GEOMETRY UNIT TEST

1. An object in which a part of the object is the same shape as the object is called _____.
2. An object in which smaller and smaller replicas of the object can repeatedly be found within it is called a _____.
3. _____ is given credit for inventing fractal geometry.
4. Name a field in which fractal geometry has applications.
5. The process of repeating the same procedure over and over again is called _____.
6. If a Koch snowflake has length of 9 inches on each side in level 2, what will be a side length in level 3?
7. If a fractal snowflake has Y sides, how many sides will it have on the next level?
8. If the snowflake perimeter is 27 on level 4, what is its perimeter on level 5?
9. With each new level of the Sierpinski triangle, how many times does the number of unshaded triangles increase form the previous level?
10. If the unshaded area of the Sierpinski triangle is 36 square inches, then the unshaded area on the next level will be _____.
11. The chaos game randomly selects from how many points?
12. The points generated in the chaos game never fall in what part of the Sierpinksi triangle?

13. The idea that very small changes in data can greatly effect the outcome is an example of _____.
14. As the unit of measurement to compute the length of a coastline decreases, the calculated coastline length will _____.
15. $\sqrt{-1}$ =
16. $(5 + 3i)^2$ =
17. $(6 - 3i) + (-2 + 8i)$ =
18. $(2 - i)(4 + i)$ =
19. Find the magnitude of $1 + 2i$
20. The Mandelbrot set is symmetric with respect to what axis?
21. If the magnitude of an iterated test point ever gets larger than _____, the test point is not in the Mandelbrot set.
22. The Mandelbrot set is confined to a region within _____ units from the origin.

If each of these test points is iterated to see if it is in the Mandelbrot set, what is the length of the third magnitude?

23. $-.5 + .2i$
24. $-.4 + 0i$
25. What are the 2 magnitudes that the test point $-1 + 0i$ alternates between?

Appendix C

Answer Keys

ANSWER KEYS

Quiz 1 - Introduction

1. Self Similar
2. Fractal
3. False
4. Mandelbrot

Quiz 2 - The Koch Snowflake

1. Koch
2. Two
3. D
4. True

Quiz 3 - The Sierpinski Triangle

1. False
2. Three
3. 3/4
4. Sierpinski

Quiz 4 - Chaos Game and Coastline Perimeter

1. True
2. Chaos
3. The Sierpinski Triangle
4. Three

Quiz 5 - Complex Numbers

1. 18 - i
2. 7 + 1.6i
3. 2 + 1.9i
4. -2.31 + 5.2i
5. 1.82

Quiz 6 - The Mandelbrot Set

1. Twenty
2. Two
3. False
4. 1

Fractal Geometry Test

1. Self Similar
2. Fractal
3. Mandelbrot
4. Medicine, Art, Weather, or Data Compression
5. Iteration
6. 3 inches
7. -4y
8. 36
9. 3 times
10. 27 sq. in.

11. 3 points
12. The shaded region
13. Chaos
14. Increase
15. i
16. 16 + 30i
17. 4 + 5i
18. 9 - 2i
19. 2.24
20. X axis
21. 2
22. 2 units
23. .46
24. .34
25. 0 and 1

Appendix D

Computer Programs

```
10 REM  RANDOM PATH GENERATOR   BY IRA FINE
15 SCREEN 12
20 CLS
23 RANDOMIZE TIMER
25 x = 320
27 y = 240
30 WHILE 5 = 5
40 cx = INT(RND(1) * 3)
50 cy = INT(RND(1) * 3)
51 IF (cx = 0) THEN x = x - 1
52 IF (cx = 1) THEN x = x + 0
53 IF (cx = 2) THEN x = x + 1
54 IF (cy = 0) THEN y = y - 1
55 IF (cy = 1) THEN y = y + 0
56 IF (cy = 2) THEN y = y + 1
57 IF (x = -1) THEN x = 0
58 IF (x = 640) THEN x = 639
59 IF (y = -1) THEN y = 0
60 IF (y = 480) THEN y = 479
63 PSET (x, y),15: PSET (640 - x, y),15
70 WEND
```

```
10 REM  THE CHAOS GAME AND SIERPINSKI TRIANGLE    BY I. FINE
15 CLS
20 RANDOMIZE TIMER
30 SCREEN 12
40 x = 200: y = 300
50 c = INT(RND(1) * 3)
60 IF c = 0 THEN 100 ELSE 180
100 x = (x + 43) / 2: y = (y + 479) / 2
120 PSET (x, y),15
130 GOTO 50
180 IF c = 1 THEN 200 ELSE 300
200 x = (x + 320) / 2: y = y / 2
210 PSET (x, y),15
220 GOTO 50
300 x = (x + 597) / 2: y = (y + 479) / 2
310 PSET (x, y),15
320 GOTO 50
```

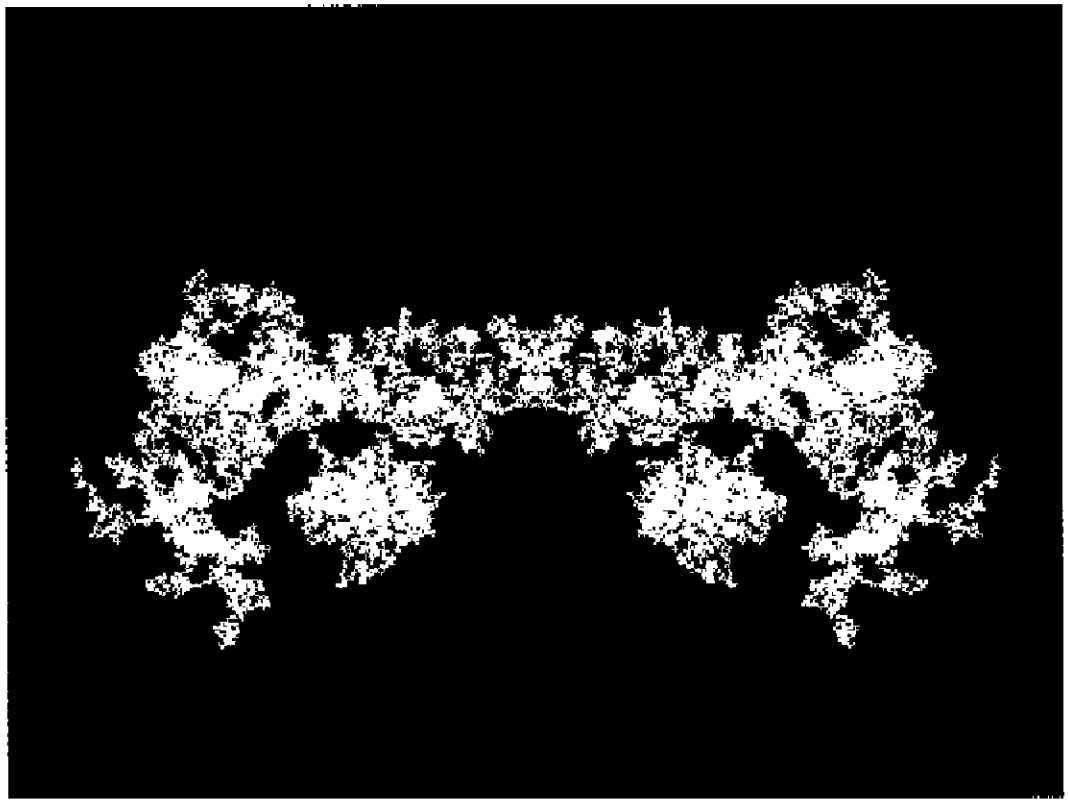
```
10 REM  MANDELBROT SET GENERATOR    BY IRA FINE
15 CLS
20 SCREEN 12
35 tb = -1.2
30 FOR b = 479 TO 240 STEP -1
32 tb = tb + .005
34 ta = -2.3
40 FOR a = 0 TO 639
42 ta = ta + .005
44 x = 0
45 y = 0
47 FOR it = 1 TO 25
49 nx = x ^ 2 - y ^ 2 + ta
51 y = 2 * x * y + tb
52 x = nx
54 mag = x ^ 2 + y ^ 2
56 IF (mag > 4) THEN GOTO 63
59 NEXT it
60 PSET (a, b), 15: PSET (a, 480 - b), 15
61 GOTO 68
63 PSET (a, b), 1: PSET (a, 480 - b), 1
68 NEXT a
70 NEXT b
80 GOTO 80
```

```
10 REM  ZOOMED IN COLOR FRACTAL   BY IRA FINE
15 CLS
20 SCREEN 12
35 tb = .3840698
30 FOR b = 0 TO 479
32 tb = tb + .000001422476#
34 ta = -1.25405058#
40 FOR a = 0 TO 639
42 ta = ta + .000001422476#
44 x = 0
45 y = 0
47 FOR it = 1 TO 300
49 nx = x ^ 2 - y ^ 2 + ta
51 y = 2 * x * y + tb
52 x = nx
54 mag = x ^ 2 + y ^ 2
56 IF (mag > 4) THEN GOTO 61
59 NEXT it
60 GOTO 68
61 c = it MOD 4 + 1
62 IF (c = 3) THEN c = 14
65 IF (it > 80) THEN c = 5
66 PSET (a, b), c
68 NEXT a
70 NEXT b
80 GOTO 80
```

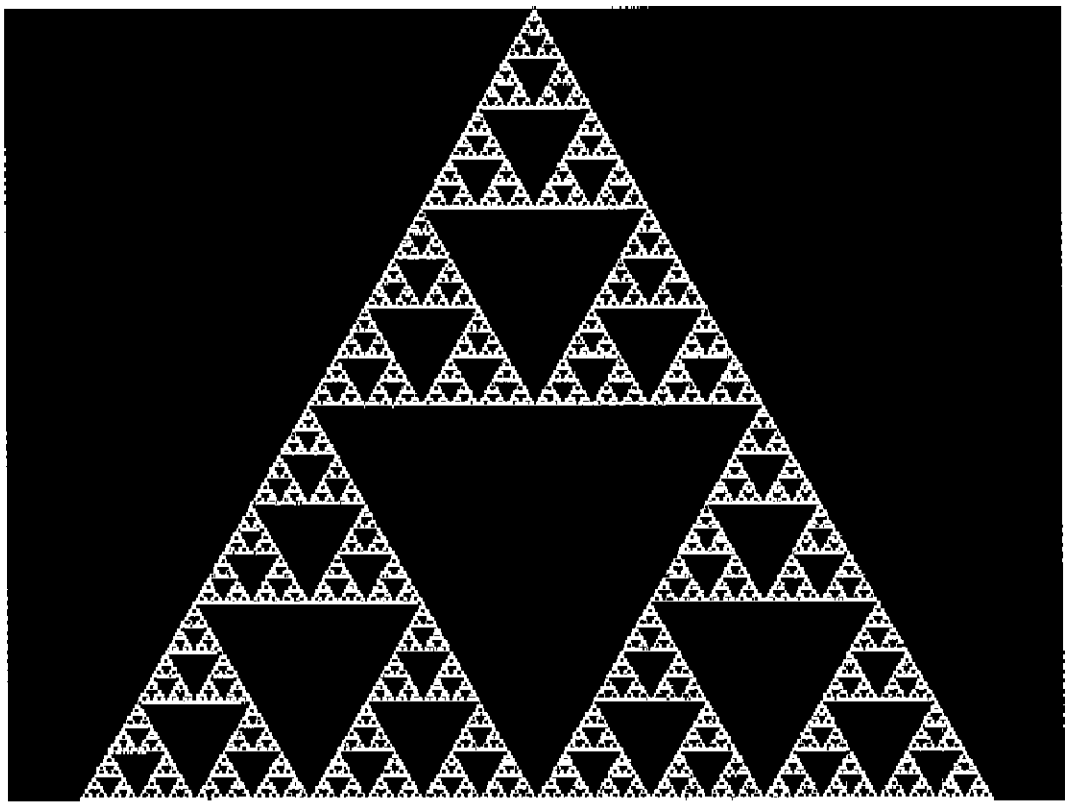
Appendix E

Computer Printouts

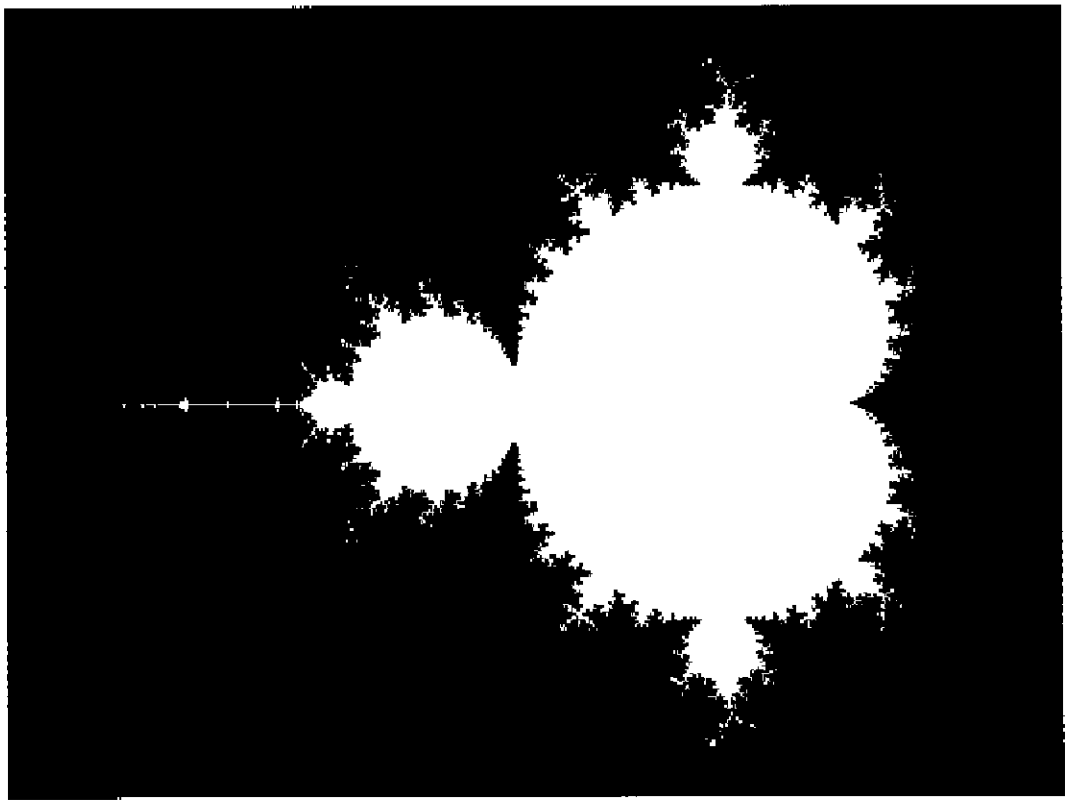
RANDOM IMAGE GENERATOR



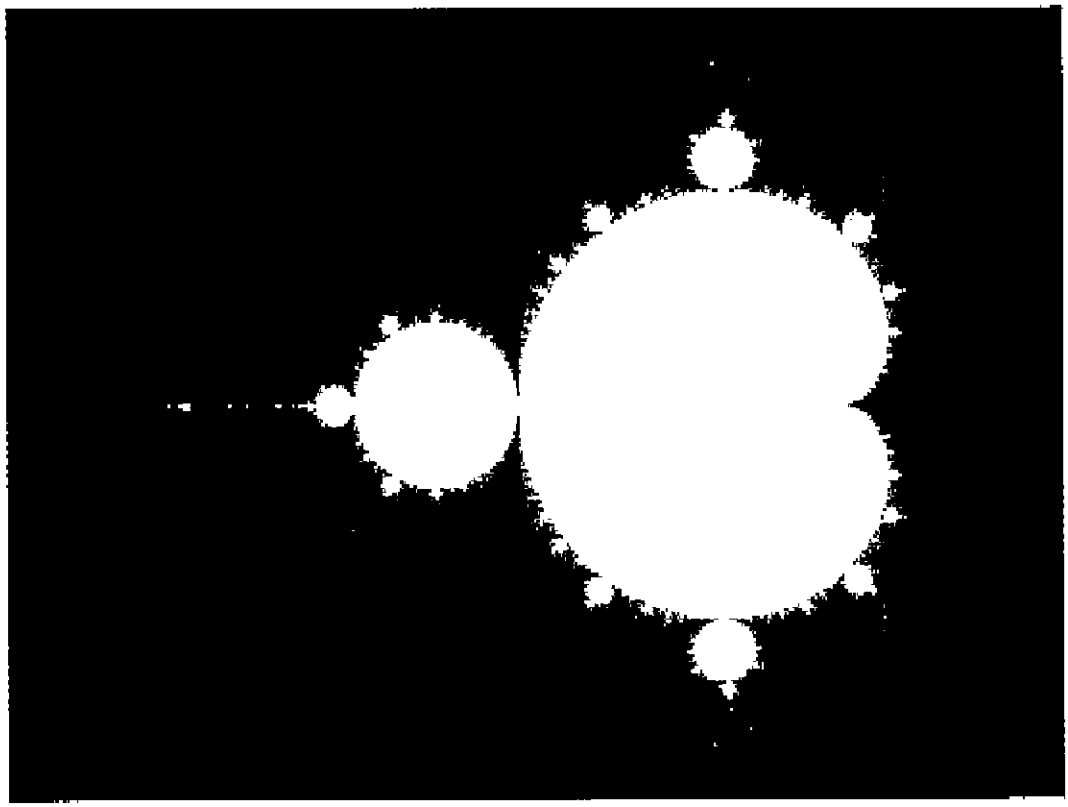
SIERPINSKI'S TRIANGLE



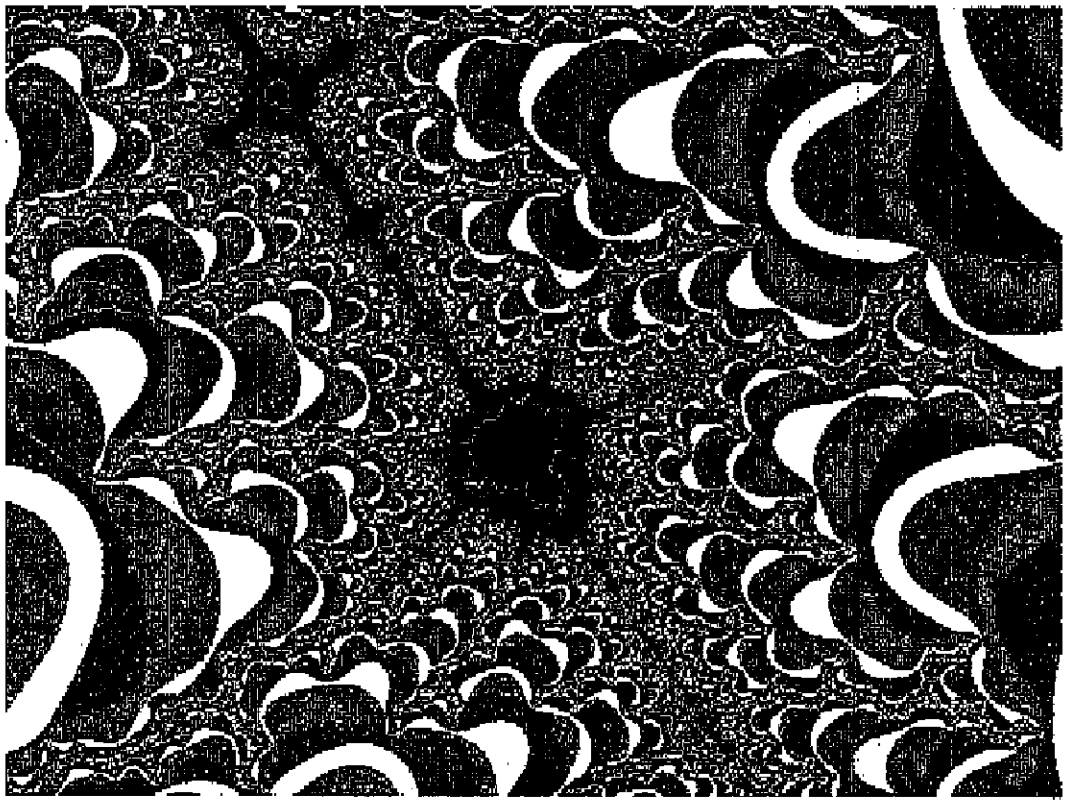
MANDELBROT SET - 25 ITERATIONS



MANDELBROT SET - 100 ITERATIONS



ZOOMED-IN COLOR FRACTAL



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