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Student understanding of Taylor series expansions in statistical mechanics

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One goal of physics instruction is to have students learn to make physical meaning of specific mathematical expressions, concepts, and procedures in different physical settings. As part of research investigating student learning in statistical physics, we are developing curriculum materials that guide students through a derivation of the Boltzmann factor using a Taylor series expansion of entropy. Using results from written surveys, classroom observations, and both individual think-aloud and teaching interviews, we present evidence that many students can recognize and interpret series expansions, but they often lack fluency in creating and using a Taylor series appropriately, despite previous exposures in both calculus and physics courses.

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I. INTRODUCTION

Over the past several decades, much work has been done investigating ways in which students understand and learn physics (see Refs. [1,2]). Some of this work has studied student understanding in particular content areas including classical mechanics [3–6], electrodynamics [7–11], optics [12,13], thermal physics [14–24], and quantum mechanics [25–27]. Other studies have focused on students' abilities to transfer knowledge and understanding between domains: either between subfields of physics [28,29] or across disciplines such as mathematics [30–33]. Most studies have been conducted with introductory students, but researchers have become increasingly interested in intermediate and upper-division undergraduate populations [8,11,18–22,25,34–44]. These populations have the potential to be particularly fascinating as they provide a glimpse of "journeyman physicists" on their way toward expertise [45].

We are presently involved in an ongoing multi-institutional investigation into students' understanding of topics related to thermal physics. We are primarily interested in identifying specific difficulties that upper-division students have with topics in and related to thermal physics and addressing these difficulties via guided-inquiry worksheet activities (i.e., tutorials) as supplements to and/or replacements for lecture-based instruction [46]. This work has led to additional investigations of student use (and understanding) of various mathematical techniques within thermal physics classes [17,22,43,44]. In this paper we report preliminary results of an investigation into students' use of Taylor series expansion to derive and understand the Boltzmann factor as a component of probability in the canonical ensemble. While this is a narrow topic, we feel that it serves as an excellent example of student knowledge transfer from mathematics and previous physics courses to more advanced topics. This topic is also particularly interesting because an understanding of Taylor series expansion is, in fact, necessary to understand the physical implications of the final mathematical expression: $\mathrm{Prob}(E) = Z^{-1}\exp[-E/kT]$, where Z is the canonical partition function. In this way, the mathematical tool (Taylor series) is not only a means to an end but a vital component to understanding the physics of the canonical ensemble.

We have previously reported various student difficulties using the Boltzmann factor in appropriate contexts as well as efforts to address these difficulties [21]. Our main effort has been the creation of a tutorial that leads students through a derivation of the Boltzmann factor and the canonical partition function [50]. As part of the derivation, students need to produce a Taylor series expansion of entropy as a function of energy.

The Taylor series expansion of a function $f(x)$ centered at a given value, $x = a$, is a power series in which each coefficient is related to a derivative of $f(x)$ with respect to x. The generic form of the Taylor series of $f(x)$ centered at $x = a$ is

$$f(x) = f(a) + \frac{df}{dx}\bigg|_a (x-a) + \frac{1}{2}\frac{d^2f}{dx^2}\bigg|_a (x-a)^2 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^nf}{dx^n}\bigg|_a (x-a)^n. \quad (1)$$

The series is often truncated by choosing a finite upper limit for the summation based on the specific context.

The Maclaurin series is a special case, generated by setting $a = 0$ in Eq. (1).

The Taylor series expansion is used ubiquitously throughout physics to help solve problems in a tractable way. It is the mathematical root of several well-known formulas across physics, ranging from one-dimensional, constant acceleration kinematics,

$$x(t) = x_0 + v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2, \tag{2}$$

through electricity and magnetism and quantum mechanics.

Other common uses of Taylor series expansions include numerical computations, evaluations of definite integrals and/or indeterminate limits, and approximations [52]. Approximations are particularly useful in physics at times when a solution in its exact form is unnecessary or too difficult to obtain; in situations where information is known about various derivatives of a function at a specific point, but nothing more is known about the function itself; or in situations in which one is investigating sufficiently small fluctuations about (or changes to) an average value. Here we are interested in students' understanding of these uses in the context of expanding the entropy of a relatively large thermodynamic system (a "thermal reservoir") as a function of the energy of a much smaller system with which it is in thermal equilibrium [53].

Relatively few studies exist within the mathematics education literature pertaining to students' understanding and use of Taylor series, the majority of which focus on conceptions related to series convergence [54–62]. Moreover, most of these articles (similar to this one) present the results of studies in which the authors investigate Taylor series as part of a broader project and not as a main focus [59]. However, a dominant theme emerging from many articles is the difficulty of synthesizing many previously learned calculus concepts to generate a robust understanding of Taylor series. Martin examines the differences between expert and novice approaches to tasks related to Taylor series convergence and reports that a "graphical understanding of Taylor series may be the single most notable effect separating novices from experts [63]." Habre also focuses on graphical representations and found that visual reasoning of Taylor series convergence may be possible even for students with poor mathematical backgrounds and who may not be able to reason about convergence analytically [58].

Recently, Champney and Kuo presented a case study of a single sophomore physics major using graphical images to reason about series truncation and the usefulness of the resulting truncation in both a purely mathematical context as well as that of a simple pendulum [61]. Their moment-by-moment analysis of an interview provides evidence for this student's evolving understanding of Taylor series and how student-generated graphical representations of a function and its series approximation greatly enhanced this understanding. One difficulty observed throughout the interview was the student's apparent lack of understanding of the importance of the center point [in their case $a = 0$ in Eq. (1)]. The student was prone to examine the behavior of the Taylor series approximation as the argument of the function got further from, rather than closer to, the center [61]. However, they did not ask the student to consider Taylor series in which the expansion point was nonzero (and would, therefore, appear explicitly in the series), so it is unclear what effect this complexity may have on student understanding.

Given the previous work that has been done and our focus on student understanding of thermal physics, we are interested in three main questions:

(1) How familiar are students with Taylor series expansions (either in the context of thermal physics or in other math or physics domains)?
(2) To what extent can students graphically interpret a Taylor series? That is, when given a graph of a function and a generic Taylor series expansion of that function, how well do students correctly connect the algebraic elements of the series with their graphical equivalents?
(3) Can students generate the Taylor series expansion of entropy as a function of energy that is used to derive the Boltzmann factor? If not, what difficulties do they exhibit and how much guidance do they need?

Our research questions differ considerably from those present in the mathematics literature. The commonality across all three mathematical studies mentioned above is the comparison between graphical representations [58,59,61]. Students (and experts in Martin's study [59]) were asked to compare the graph of a function to the graph of a truncated Taylor series approximating it and comment on the validity of the approximation. We are interested in students' abilities to connect the algebraic forms of individual terms in the Taylor series to specific features of a graph of the function (e.g., the slope at a given point) rather than their abilities to compare two graphs. We are also interested in student use of Taylor series in a specific physical context, that of the canonical ensemble. This physical context is the motivation for our study and guides our research design and methods.

To answer our research questions as completely as possible we collected data using written surveys, classroom observations, and student interviews. Our objective was to identify and document specific difficulties that students displayed while engaging with Taylor series expansions. As such, we emphasize the description of students' actions and utterances over our interpretations, and we recognize that any descriptions of students' ideas are our own assumptions based on the data [64]. We often analyze the data holistically to identify trends across students and data sets. Details of our data collection and analysis are contained in the following sections.

We have two main findings from this work, one focused on student understanding and one related to pedagogical strategy. Our primary result is that data from written surveys and clinical interviews suggest that many students are familiar with the Taylor series but may not use it fluently in physical contexts. A secondary result, gleaned from teaching interviews and several years of tutorial implementation and observation, is that a pretutorial homework assignment can provide students with a necessary opportunity to refresh their memory of what exactly a Taylor series is and how to use it to model physical contexts.

II. RESEARCH METHODS

Data for this study were gathered during four consecutive years in an upper-division undergraduate statistical mechanics course (Stat Mech) at a public research university in the northeastern United States. The course enrolls approximately 8–12 students each spring semester, and the population under investigation was composed primarily of senior undergraduate physics majors and a few physics graduate students. Stat Mech meets for three 50-minute periods each week. Most instruction uses lectures, but tutorials are used in place of lecture for between five and seven class periods each semester. Stat Mech contains no explicit instruction on Taylor series; however, all students had previously completed at least one calculus course covering various series and summation topics (including Taylor series expansion). All undergraduate physics majors had also taken a sophomore-level mechanics course that includes the use of Taylor series in several applications, as well as a junior-level course on mathematical methods in physics. Graduate students in this study reported having previously taken both mathematics and physics courses at their undergraduate institutions that explicitly utilized Taylor series expansions. During data analysis we were sensitive to the possible impacts of variations in student preparation.

Data were collected using written surveys, student interviews, and classroom observations. Triangulation of data sources in this way allows for a deeper probe of student difficulties than a single data set by providing multiple pieces of information about individual students. Using multiple data sources also affords the opportunity to observe students' behaviors in various instructional settings. Collecting data in this manner also allowed us to monitor the ways in which students engaged with the Boltzmann factor tutorial (including their use of the Taylor series expansion). Using videotaped classroom observations and interviews, we observed whether or not students struggled when we wanted them to struggle and/or succeeded when we hoped they would succeed. Table I shows a time line of research activities over the four years of the study. In the following sections we provide a brief overview of our research instruments and the methods we used to analyze the data from each. More detailed descriptions of the research instruments are contained in Secs. III, IV, and V, where we present the data gained from each and interpret the corresponding results.

TABLE I. Research time line. An outline of the research activities that took place either before, during, or after tutorial implementation for each year of the study. "PT" refers to the Taylor series pretest, "HW" refers to a pretutorial homework assignment that students completed and used during the tutorial, and "Observations" refers to videotaped classroom observations.

Year	Before tutorial	During tutorial	After tutorial
1	PT, HW	Observations	Teaching interviews 4 weeks after
2	PT, HW	Observations	Clinical interviews 4–5 weeks after
3	PT, HW	Observations	
4	PT, HW		

A. Written surveys

The Taylor series pretest (see Sec. III) assesses students' abilities to interpret Taylor series of a function (centered at three separate values), given a graph of the function; it was given as an ungraded in-class survey before tutorial instruction in all four years of the study [65]. Students' responses to the Taylor series pretest were categorized in two ways: first by their answers chosen from the provided options (positive, negative, or zero), and second by the explanations they provided. Analyzing these explanations, we used a grounded theory approach in which the entire data corpus was examined for common trends, and all data were reexamined to group them into categories defined by these trends [66,67]. Our analysis focused on describing rather than interpreting students' explanations while defining the categories. In this way our analysis stays as true to the data as possible by limiting researcher biases and interpretations. This is consistent with Heron's identification of specific difficulties [64]. Data from the Taylor series pretest provide insight into students' prior knowledge regarding Taylor series expansions.

B. Classroom observations

Given that the focus of our data gathering and analysis was to examine student ideas regarding Taylor series expansions and to monitor students' abilities to efficiently and productively complete the Boltzmann factor tutorial, classroom data were gathered by videotaping classroom episodes (one or two each semester) of students working in small groups (2–4 students) to complete the Boltzmann factor tutorial. Segments from these classroom episodes were selected for transcription and further analysis based on the content of student discussions. Given our focus on investigating students' understanding of particular topics, our methods of gathering video data align with Erickson's description of *manifest content* approaches, in which particular classroom sessions are selected to be videotaped based on the content being discussed [68]. We chose to videotape classroom sessions in which students were engaging in the Boltzmann factor tutorial because we were primarily interested in their ideas regarding the Boltzmann factor and their use of Taylor series expansion. During each tutorial session we videotaped one or two groups. During data analysis each video was watched in its entirety, noting segments that would be interesting and useful for further analysis. These segments were then transcribed along with researcher notes and impressions. Student quotations included in the following sections were often selected because they were novel and/or indicative of opinions expressed by the group. Several students made comments and statements that indicated difficulties that were not expected and have not been previously documented. Data do not exist to verify the pervasiveness of these difficulties, but we feel their existence is noteworthy. In cases where more than one student displayed a similar difficulty, we have included multiple quotes to allow the reader to evaluate both similarities and differences.

During analysis of classroom observations, attention was paid more to the physics content expressed during students' discussions than the broader social interactions evident within the video. While the data obtained could certainly be analyzed using existing literature on gestures and interpersonal interactions (see Ref. [69] and references therein), the focus of this overarching project, and our interest in the data, lies in students' ideas regarding the conceptual and mathematical content of the Boltzmann factor tutorial and students' abilities to negotiate tutorial prompts in an efficient and productive manner. For our purposes a "productive" student interaction is one in which they discuss topics related to the tutorial in a way that helps them progress through the tutorial tasks while seeming to gain an appropriate understanding of those topics (discussing relevant concepts, synthesizing information, engaging with the connections between the mathematics and the physics, etc.). This corresponds most closely to *productive disciplinary engagement* [70]. An "efficient" interaction is one that allows the students to complete the tutorial within the intended 50-minute class period. In some respects the categorization of student interactions is done with an eye toward the end justifying the means: an interaction cannot necessarily be considered productive or efficient without knowing the conversations that take place both before and after that interaction. We paid particular attention to students' use of completed pretutorial homework assignments to see if these assignments helped improve their efficiency during the tutorial.

C. Student interviews

In an effort to delve further into students' ideas regarding concepts related to the Boltzmann factor tutorial, we conducted interviews with students either individually or in pairs [71]. In order to solicit interview participants, all students in Stat Mech were invited to participate, and all interested students were interviewed. In this way, interviewed students were self-selected, but they represented a broad spectrum of ability in Stat Mech during years in which interviews were conducted. Interviews were conducted in a think-aloud style in which students were encouraged to verbalize their thought processes while completing interview tasks [72]. We conducted two rounds of interviews in two different years, each with a different purpose. In year 1 of the study we conducted interviews with four students in the style of a teaching experiment to test the instructional strategies used within the Boltzmann factor tutorial [73,74]. It should be noted that the teaching experiments (or "teaching interviews") were not conducted to determine students' understanding of the Boltzmann factor. Instead, we wanted to examine how well students could complete instructional tasks based on their previous knowledge related to the Boltzmann factor and Taylor series expansion. As a result of these teaching interviews being conducted after the initial tutorial implementation (in year 1 of the study), they could also inform tutorial revisions and improve instruction in subsequent years. According to Steffe and Thompson [73], "a teaching experiment involves a sequence of teaching episodes ... [including] a teaching agent, one or more students, a witness of the teaching episodes, and a method of recording what transpires during the episode." During our interviews, one researcher alternated roles as both teaching agent and witness. In a sense, the tutorial activities used during the interview may also be seen as a teaching agent because they contain instructions for students to perform tasks, and students interacted with the document in an intellectual manner. One of the unique aspects of a teaching experiment as an approach to interview procedures is that "it is an acceptable outcome ... for students to modify their thinking" during the course of the interview [74]. Our objectives during these interviews were twofold: to see how successful students would be at working through tutorial tasks and, when difficulties arose, to see what interventions helped students succeed. These interviews were a valuable source of data on students' understanding of content presented within the tutorial (including Taylor series expansion). Field notes were taken during the interviews, and students' written work was collected afterward and examined in a manner consistent with our treatment of students' responses to written questions.

During other interview tasks (in year 2 of the study), we were interested in investigating students' ideas about Taylor series expansions without influencing them. With that in mind we used a clinical interviewing technique similar to those described by Piaget and Inhelder to examine five students' ideas about these topics several weeks after tutorial instruction [74,75]. This allowed us to examine students' understanding of topics related to the Boltzmann factor tutorial more deeply than we could using either written surveys or classroom observations. In the clinical interview setting the students were asked a series of specific questions related to Taylor series expansions regarding their use in physics, truncation of the series, etc. Based on students' responses to the interview prompts, additional questions were asked to further probe their thought processes. Three individual interviews and one two-student interview were conducted several weeks after the second implementation of the Boltzmann factor tutorial.

As with written surveys, we used a grounded theory approach for analyzing all video data (teaching interviews, clinical interviews, and classrooms observations) in an attempt to find interesting and common trends [66,67]. However, with a small data set (about five videos for each type of interview or classroom observation) trends were not always apparent. As such, many videos are treated as case studies, and emphasis is placed on describing the data before interpreting them. We strive to provide a description of the student population that may be found in an upper-division statistical mechanics course [76]. Both the interview protocol used during the clinical interviews and the tutorial activity used during the teaching experiments are provided in Appendixes A and B.

III. STUDENT INTERPRETATION OF A TAYLOR SERIES EXPANSION

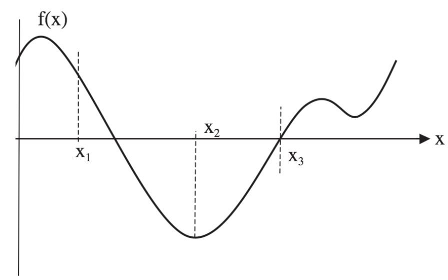
Studies have shown that upper-division physics students often struggle with mathematical concepts independent of the physical contexts in which they may be applied [33]. We anticipated that generating a Taylor series expansion would be more challenging to students than interpreting a generic series [59]. With that in mind we chose to first test students' abilities to interpret the terms of a generic Taylor series expansion (graphically) in a purely mathematical context.

In the Taylor series pretest, students are instructed to interpret the terms of a Taylor series expansion based on a given graph of a generating function $f(x)$ shown in Fig. 1. The truncated Taylor series expansion centered at the point $x = x_1$ is given as

$$f(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2. \tag{3}$$

Students are asked to determine whether each of the quantities a_1, b_1, and c_1 is positive, negative, or zero and to explain their reasoning based on the graph (with x_1 clearly marked). The same question is then asked for Taylor series centered at x_2 and x_3 on the graph. The correct answers require students to recognize that a_i is the value of the function at x_i, b_i is the slope of the function at x_i (i.e., the first derivative), and c_i is proportional to the concavity (corresponding to the second derivative) at x_i.

Below is a sketch of the function $f(x)$.



The function $f(x)$ is expanded in a Taylor series about $x = x_1$, $x = x_2$, and $x = x_3$. For each point, state whether the given coefficients are *positive*, *negative*, *zero*, or *not determinable with the given information*. Explain how you determined your answers using words and/or sketches on the graph above.

FIG. 1. Taylor series pretest. Students were given equations similar to Eq. (3) for series centered at x_1, x_2, and x_3 to show the algebraic meanings of the coefficients a_i, b_i, and c_i.

A main component of the Taylor series pretest is the consideration of several different series for the same generating function that are centered at different points. Martin reports that both experts and novices have the capacity to reason about recentering a Taylor series, but that experts do so much more fluently [77]. The Taylor series pretest also deliberately avoids using a Maclaurin series, as the physical context in which we expect students to later generate a series requires a nonzero center.

Two examples of correct student responses to the Taylor series pretest are shown in Table II. These quotes were chosen as representative of all correct student responses. It should be noted that both students are considered completely correct even though they gave different answers for coefficient c_1: the concavity of the graph at x_1 is sufficiently small that we considered both "negative" and "zero" to be correct if students provided appropriate

TABLE II. Examples of correct student responses on the Taylor series pretest. (All student names are pseudonyms.)

Student 1 (Paul)	
a_1: (+)	"a is based on $f(x)$."
b_1: $f' = (-)$	"b, c are the first and second derivatives, respectively."
c_1: $f'' = (-)$	
Student 2 (Jonah)	
a_1: positive	"It is above the $f(x)$ 0 axis."
b_1: negative	"It is a negative slope."
c_1: zero	"There is no concavity."

reasoning for their responses (as Paul and Jonah did). In the four years that this question was given before tutorial instruction, 20 out of 35 students correctly determined all the signs of the various quantities and gave appropriate reasons (another five students gave correct answers without sufficient explanations) [78]. These data suggest that a majority of the students in Stat Mech are familiar with the meanings of the various terms in the Taylor series.

No consistent trends are evident in incorrect student responses to the Taylor series pretest, which is not surprising in a small data set (only 10 out of 35 students were considered to be incorrect). However, several notable observations may serve as evidence of the existence of many different difficulties related to the interpretation of Taylor series. Two students related all three of the a values to the "y-intercept" or "$f(0)$." One of these students also indicated that b_1 and c_1 would be, respectively, the first and second derivatives of $f(x)$ evaluated at $x = 0$: "$\frac{df(0)}{dx} = b_1$; $\frac{d''f(0)}{dx} = 2c_1$," and stated that the "same line of reasoning can be used for the remainder of these functions" (presumably to determine b_2, c_2, b_3, and c_3). However, this student did not give an explicit answer for the sign of any of the listed coefficients. This focus on the function at $x = 0$ may be due to the Maclaurin series being used in the vast majority of Taylor series tasks during typical calculus instruction [79]. Habre also reports students' increased familiarity with Maclaurin over the more general Taylor series and students' lack of understanding of the importance of being able to center the Taylor series at any value [58]. The Maclaurin series is a reasonable instructional starting point, but overemphasis may detract from the value of seeing the center point as an arbitrary variable in a Taylor series.

Three students gave responses indicating that the signs of a, b, and c would alternate at each point but differed in their reasoning and the pattern of alternating. One student (correctly) indicated that a represents the value of the function at each point and wrote the (incorrect) Taylor series for both the sine and cosine functions as evidence of his reasoning:

$$\sin(x) = a + \frac{ax^2}{2!} - \frac{ax^4}{4!} + \frac{ax^6}{6!};$$
$$\cos(x) = ax - \frac{ax^3}{3!} + \frac{ax^5}{5!}.$$

Another student indicated that a represents the slope of the function at a given point and that the expansion always alternates signs. The third student offered no explanation for his responses but supplied alternating-sign answers similar to the others. It is unclear why these students assume the signs of the coefficients must alternate, but perhaps the first student's allusion to the Taylor series of sine and cosine provides evidence that these touchstone examples are more accessible to these students than appropriate applications to generic functions.

The remaining five students provided incorrect responses that did not share commonalities with any other responses (two of which were mostly blank and void of any explanation). One student seemed to try to describe local features of the function around each point, stating that the function "starts above zero and looks like an inverted parabola in the region [of $x = x_1$]." His answers for the x_1 coefficients were consistent with a parabola of negative concavity having a maximum value at x_1 (i.e., $a_1 > 0$, $b_1 = 0$, and $c_1 < 0$). Another student wrote that, "for $x = x_1$, $x - x_1 = 0$" as a justification for her answers that $b_1 = c_1 = 0$ (as well as b_2, c_2, b_3, and c_3 being zero). This may stem from the idea that the b and c coefficients are being multiplied by a value that is exactly zero at the center of the series [80]. The final student correctly determined all b and c coefficients "by thinking about 1st, 2nd derivatives and how they behave at maxima, minima, and inflection points," but incorrectly determined all three a values. It is unclear how this student determined the signs for the a values as no more-explicit explanation is provided. The myriad incorrect responses given on the Taylor series pretest suggest that this topic warrants in-depth future investigation.

IV. STUDENT USE OF TAYLOR SERIES IN CONTEXT

While studying student understanding of general Taylor series representations is interesting in its own right, we are primarily interested in students' abilities to use Taylor series to reason about a particular physical context: that of the canonical ensemble that is presented in our Boltzmann factor tutorial. Our data for this investigation come from field notes of individual students participating in teaching interviews and videotaped classroom observations of small groups of students working through the Boltzmann factor tutorial. In the following sections we present our observations and comment on instructional implications.

A. Student difficulties generating Taylor series during teaching interviews

Teaching interviews were conducted with four students in year 1 of the study to determine their abilities to complete the tutorial tasks (see Appendix A for the teaching activity used during the interview). In the tutorial, students are asked to complete the derivation of the Boltzmann factor, which involves a small system in thermal contact with a large reservoir (e.g., Ref. [51]). Because the system is so small relative to the reservoir, the derivation requires students to generate a Taylor series expansion of the entropy of the reservoir ($S_{\rm res}$) as a function of its energy ($E_{\rm res}$). The intended final result is an expression for the entropy of the *reservoir* as a function of the energy of the *system*: $S_{\rm res}(E_{\rm syst})$. The desirability of this result is that the derivative of the entropy of the reservoir with respect to its own energy is related to its (constant) temperature ($\partial S_{\mathrm{res}}/\partial E_{\mathrm{res}} = T^{-1}$), and that the energy of the system is the quantity of interest in the canonical ensemble. Interviewed students were presented with this physical situation and asked to write a Taylor series expansion of the entropy of the reservoir as a function of its energy about the (fixed) total energy ($E_{\mathrm{tot}} = E_{\mathrm{res}} + E_{\mathrm{syst}}$):

$$\begin{aligned} S_{\mathrm{res}}(E_{\mathrm{res}}) &= S_{\mathrm{res}}(E_{\mathrm{tot}}) + \frac{\partial S_{\mathrm{res}}}{\partial E_{\mathrm{res}}}\bigg|_{E_{\mathrm{tot}}} (E_{\mathrm{res}} - E_{\mathrm{tot}}) + \cdots \\ &= S_{\mathrm{res}}(E_{\mathrm{tot}}) - \frac{1}{T} E_{\mathrm{syst}} + O(E_{\mathrm{syst}}^2), \end{aligned} \tag{4}$$

where the simplification in the second line comes from the considerations that (a) the system and the reservoir have the same constant temperature: because of the constant temperature of the canonical ensemble, higher-order derivatives of entropy with respect to energy vanish; (b) the energy of the system can be written as $E_{\mathrm{syst}} = E_{\mathrm{tot}} - E_{\mathrm{res}}$; and (c) $E_{\mathrm{syst}} \ll E_{\mathrm{res}}$. The second line of Eq. (4) is a very valuable result, as it relates the entropy of the reservoir to the energy of the system.

During the four teaching interviews, only one student (Joel) succeeded in spontaneously generating a Taylor series expansion of reservoir entropy as a function of energy [as in Eq. (4)]; however, other portions of Joel's interview suggest that his success was a result of memorizing the derivation of the Boltzmann factor in the course textbook [51] rather than evidence of thorough comprehension of the Taylor series [21,81]. Two other students were able to generate the appropriate expansion only after they were given the following expression for a Taylor series expansion of entropy as a function of energy about the value $E = E_0$:

$$S(E) = S(E_0) + \frac{\partial S}{\partial E}\bigg|_{E_0} (E - E_0) + \frac{1}{2!}\frac{\partial^2 S}{\partial E^2}\bigg|_{E_0} (E - E_0)^2 + \cdots. \tag{5}$$

The final student was also given this expression but was unable to connect it with the specific physical situation without explicit instruction from the interviewer.

These results indicate that student understanding of when and how to use a Taylor series expansion (a crucial part of the derivation used in the Boltzmann factor tutorial) should not be taken for granted. When combined with the data from the Taylor series pretest discussed above, these results indicate that students may be able to interpret and apply a Taylor series that is given to them, but not be able to generate an appropriate expansion in a novel context. These difficulties were not entirely unexpected.

B. Instructional intervention: Pretutorial homework

One instructional strategy that we incorporated into our curriculum development is the use of what we call "pretutorial homework." We have seen, at the upper division in particular, that much detailed prerequisite knowledge—both mathematical and physical—must be readily accessible in order for students to make appropriate progress through the tutorial in the allotted time. One way we have found to address this issue is to assign homework to be completed prior to the tutorial, in which students engage with the prerequisite knowledge as their schedules permit. The use of similar homework activities has been shown to be quite effective in preparing students for class (e.g., the WarmUp questions used in Just-In-Time Teaching [82]), but our use of homework as preparation for students to engage in a guided-inquiry tutorial deviates from the typical pretest-tutorial-homework sequence employed at the introductory level [83]. Future publications will detail the benefits we find in including pretutorial homework in the tutorial sequence for upper-division topics [84].

We developed a two-question pretutorial homework assignment for the Boltzmann factor tutorial [85]. In the first question, students are asked to write a Taylor series expansion of entropy as a function of energy (including no more than five terms) about the value $E = E_0$ [see Eq. (5)]. This gives students the opportunity to look up the generic form of the Taylor series [cf. Eq. (1)] and apply it to the given situation. They bring that expansion with them to class as a resource to use while deriving Eq. (4) during the tutorial.

In the second homework question, students are asked to give an interpretation of how each of the terms in the Taylor series relates to a given graph of S versus E (see Fig. 2). This question encourages students to think about the meaning of the terms in their Taylor series rather than merely copying down abstract symbols. This also provides students who did poorly on the Taylor series pretest with an opportunity to address some of the issues in their understanding of these terms. The pretutorial homework was assigned to students in all four years of the study.

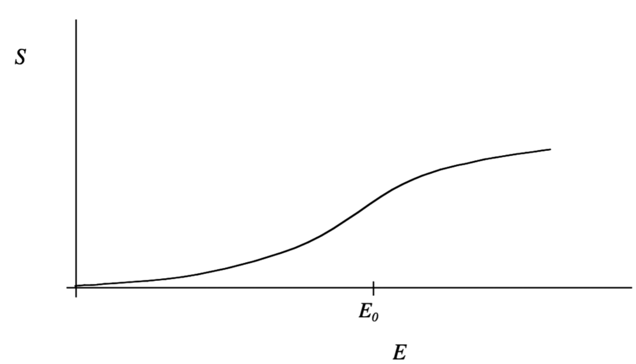


FIG. 2. Graph used in the pretutorial homework assignment. Students were asked to give a graphical interpretation of each of the terms in their Taylor series expansion of entropy as a function of energy about $E = E_0$ [see Eq. (5)].

C. Student difficulties with Taylor series during classroom activities

One unexpected difficulty observed during the tutorial session in year 2 is that two students (Sam and Bill, both senior undergraduate physics majors) did not correctly construct the Taylor series expansion asked for in the pretutorial homework assignment. Instead of constructing the appropriate expansion as seen in Eq. (5), they used the terms "S_1" and "S_2" in place of the "$(E - E_0)$" and "$(E - E_0)^2$" terms, respectively, (i.e., $S = S_0 + S_1 S' + \frac{1}{2} S_2 S'' + \cdots$, where S_0, S_1, and S_2 were said to be constants). This expansion not only made it impossible for the students to obtain an expression for entropy as a function of energy, but also prohibited them from obtaining a dimensionally accurate expression for entropy at all. These students did, however, recognize that their expression lacked an energy term, and once an instructor intervened to discuss the appropriate form of the Taylor series with them, they were able to use it correctly to complete the derivation of the Boltzmann factor.

Students in other groups had greater success completing the portion of the Boltzmann factor tutorial that required generating an appropriate Taylor series expansion. A group in year 2 correctly completed the task with very little instructor intervention [86]. These students referred to their (correctly) completed pretutorial homework assignments several times during the class: to clarify the desired task within the tutorial, to determine the center point about which the expansion should be taken, and to help interpret their resulting expression.

A group of students in year 3 also completed the Taylor series task with relative ease. They referred to the pretutorial homework assignment to help them begin to generate an appropriate Taylor series. Most of their subsequent discussion focused on details of the series and how to relate E_{res}, E_{syst}, and E_{tot} to obtain an expression for $S_{\text{res}}(E_{\text{syst}})$. The only substantial instructor intervention resulted in the students changing their notation of the derivative from "S' " to "$\partial S/\partial E$." Immediately after making this change the students recognized that this derivative was equivalent to T^{-1} and thus a constant. This "prime" notation for a derivative is often used as a shorthand notation in the undergraduate physics curriculum. However, its use in this context by these students (as well as Sam and Bill mentioned above) hindered their ability to interpret physically the terms of their Taylor series by omitting any explicit reference to energy. This result may allude to a broader issue regarding the use of shorthand notations in relation to students' abilities to interpret mathematical expressions. The explicit partial derivative notation is likely more beneficial (perhaps essential) in the context of more cognitively challenging concepts, such as the physics studied here.

D. Student difficulties at another institution

During year 2 the Boltzmann factor tutorial was also implemented in a junior-level undergraduate thermal physics course at a comprehensive public university in the western United States. The instructor of this course reported that many students had great difficulty using the Taylor series expansion in the tutorial context even after having completed the pretutorial homework. In an effort to help the students, the instructor split the Boltzmann factor tutorial into two class periods and assigned specific study of the Taylor series between the two periods. He reported that a short lecture on the use of Taylor series expansion was necessary at the beginning of the second tutorial period to allow students to successfully complete the tutorial. This report provides further evidence that the use of the Taylor series in this context is not trivial and suggests that student difficulties in this area are not localized to our student population.

Data from our primary implementation site provide evidence that completing the pretutorial homework assignment can be helpful to students during the tutorial session and allow them to complete tutorial tasks in a productive and efficient manner. However, data from both implementation sites show that the assignment itself is not trivial, and that some students may fail to complete the task. These data also suggest that students' successes on the pretutorial homework and the tutorial itself may depend strongly on their previous exposures to Taylor series and related calculus topics. This is consistent with Habre's findings regarding students' abilities to reason graphically and/or analytically about Taylor series convergence [58]. In practice, not all students (even within a single class) will have experienced the same level and type of preparation in terms of mathematics or physics classes dealing with Taylor series. Therefore, it would be valuable to gather more data on upper-division physics students' understanding of Taylor series in an effort to design instruction that may reach students at various levels of mastery.

V. FURTHER INVESTIGATION INTO STUDENT UNDERSTANDING OF THE TAYLOR SERIES

After tutorial implementation in year 2, which included the pretutorial homework, clinical interviews were conducted with five students, four of whom had participated in the Boltzmann factor tutorial in class (see Appendix B for interview protocol). A primary purpose of the interviews was to determine how familiar the students were with Taylor series expansions, including when they are applicable and how they are used.

A. Student ideas about when to use a Taylor series

All students interviewed had a reasonable understanding of situations in which the Taylor series is an appropriate tool. All students spontaneously used terms like ''approximation'' and ''estimation'' when describing how to use a truncated Taylor series expansion, and all students were able to list one or more specific areas of physics in which Taylor series expansions are useful.

Interview prompt: What do you know about the Taylor series? That is, when I say ''Taylor series,'' what comes to mind?

Malcolm: Most useful functions can be approximated by the Taylor series.

...

Any time we're dealing with a potential that can be approximated as a harmonic potential, Taylor series approximations usually spring out of that.

Paul: It's a way to approximate any function.

Jonah: Well you can approximate sine with a Taylor series. Or cosine or any ... That's just what we use it for. Just like the power series, it's just a modification of that.

Paul: The idea of a series is to simplify something that is more complex isn't it? So the first few terms will be the most significant, and the rest will die out pretty quickly.

Kyle: Approximation of a function, truncate them, perturbations used in mechanics, quantum, stat mech.

Jayne: An expansion that we use to—I don't know if estimate is what I am looking for ... It's a good way to describe functions if you don't know what's going on at a certain point. What comes to mind? I guess $x = x_0 + v_0 t + \frac{1}{2}at^2$.

...

But now using [Taylor series] in thermo, clearly it's a little more ... in-depth, but as you get into more physics you see it more and more.

One interesting aspect of the clinical interviews is that all students at some point during the interview *spontaneously* referred to the kinematics equation [Eq. (2), with $t_0 = 0$] as an example of a Taylor series. This had been described during a Stat Mech lecture as an example of a Taylor series expansion with which everyone would be familiar (even if they had never recognized it as a Taylor series). Their acknowledgment of that kinematics equation as a Taylor series seemed to influence their responses to various interview prompts. Students often referred to the kinematics equation, and many of them discussed knowledge about higher-order derivatives as a viable means for deciding when to truncate the series [as in Eq. (2) when acceleration is constant]. Throughout the remainder of this section we present many aspects of student interviews as individual case studies. As the data show, student responses to interview questions varied widely across our small student population. This being the case, we find it useful to present data from each interview individually rather than attempting to generalize across the entire population.

B. Student ideas about series truncation

One of the primary questions the interviewees were asked about the Taylor series was, How do you know when to truncate the series? (See Appendix B.) A common response involved knowledge about the functional form of any higher-order derivatives; i.e., if one of the derivatives is constant, then all higher derivatives will be identically zero. Malcolm, a physics graduate student, used this reasoning to justify why the kinematics equation has only three terms: ''Usually acceleration's constant, so we don't have a jerk. If we had a jerk running around messing things up, we'd need more terms.'' When prompted about situations in which no information was known about the derivatives, however, Malcolm said that he would use different ''rules of thumb'' depending on the application. If only a ''ballpark estimate'' was needed, for example, only one or two terms would be necessary, but he indicated more terms would be needed as desired precision increased (e.g., to ''16 digits''). Malcolm also noted that looking close to the value about which he was expanding would require fewer terms than if he were trying to extrapolate far from the expansion point. Finally, Malcolm stated that he would examine the deviation between the Taylor series expansion and any experimental data available and keep enough terms to have a reasonable fit (although he did not specify how closely he would require the expansion to match experimental data).

The relation to experimental data was echoed by Jayne (another graduate student) who initially had trouble articulating a good rationale for truncating a series but eventually referred to different needs for different experimental tasks. Jayne cited a threshold for truncation of 3 or 4 orders of magnitude; i.e., terms that are 3–4 orders of magnitude smaller than the linear term are not necessary.

The two undergraduate physics majors who were interviewed together (Paul, who had participated in the Boltzmann factor tutorial, and Jonah, who had not) also cited constant acceleration as the reason why the kinematics equation only has three terms and knowledge of constant derivatives (derivatives or coefficients independent of the choice of the center point) as the primary reason to truncate a Taylor series. After several prods and questions about series truncation they started using estimation language to discuss the possibility of starting with a ''ballpark'' estimate and keeping terms until the results were close enough (using a guess-and-check type of method). Paul also argued that the purpose of a Taylor series is to estimate something that is more complex and that the first few terms should be the most significant while the higher-order terms die out.

C. Summary: Students may miss the big picture

All interviewed students were able to list some areas of physics in which the Taylor series might be useful beyond the kinematics equation (examples in quantum mechanics, solid state physics, statistical mechanics, etc.), but no one elaborated on exactly *how* a Taylor series would be useful in these various situations. Malcolm (quoted above) came closest by citing the use of Taylor series to approximate a potential in quantum mechanics as a harmonic potential. It is still unclear, however, what (if anything) would motivate these students to use a Taylor series expansion *spontaneously* in a given physical situation. We do not have evidence that they are able to generalize their knowledge to state the conditions under which a Taylor series is appropriate, and/or how many terms to retain. It seems as though their past experience has been limited to various instructors and texts specifying both when to use a Taylor series and how many terms to retain. These results are consistent with recent literature in mathematics education, particularly Champney and Kuo, who report on student difficulties reasoning about truncating a Taylor series and the usefulness of the resulting expression [61].

VI. VIEWING OUR FINDINGS THROUGH MODERN TRANSFER FRAMEWORKS

Our intent in this research was not to investigate transfer explicitly. The Taylor series expansion surfaced as one area where prior research on student use of mathematics in physics—ours as well as others'—would suggest that one could anticipate student difficulties.

Researchers in both mathematics and physics education research have begun to investigate learning and teaching with a transfer lens, as evidenced by a recent monograph on transfer, which included several articles from physics education research, research in undergraduate mathematics education, and psychology [87]. Indeed, given the many theoretical frameworks for transfer in the literature of various education research and learning sciences domains, a thorough analysis of our findings through the many frameworks would constitute a massive undertaking and is well beyond the scope of our current work. However, a brief exploration of how our findings can be interpreted through some of these frameworks is warranted.

One characteristic of many of these more recent frameworks for transfer is the expansion of the definition of transfer to resemble more closely what has been traditionally referred to as learning. In his description of *transfer in pieces* [88], built on diSessa's knowledge-in-pieces perspective [89] for making sense of student responses, Wagner argues that transfer involves incremental learning in different contexts that gradually extends the span of perceived applicable situations in which to apply a concept [90]. Wagner defines learning as the (re)organization of useful intuitive knowledge in ways that provide a structure for interpreting observations and systematic understanding. According to Wagner, knowledge transfer is identified as ideas, previously used in particular contexts, are introduced into new contexts. Rather than transfer being manifested as the development and use of context-independent knowledge in a particular context, transfer in pieces occurs the other way around, when the recognition of similarities across contexts leads to the abstraction of ideas from the individual contexts. Abstraction is the consequence of transfer: this is how students realize that things that seem different are actually similar in an important way.

Schwartz, Bransford, and Sears [91], building off of Bransford and Schwartz [92], distinguish two types of transfer. The first is referred to as *transfer out*, and resembles the traditional view of transfer, i.e., direct application of previously learned knowledge typically used to solve problems. Efficiency at problem solving involves rapid and accurate transfer out. They also describe *transfer in*, the flexible and often spontaneous use of prior knowledge to interpret given information in learning situations. (It could be argued that the activation of diSessa's phenomenological primitives in specific scenarios constitutes transfer in; the knowledge transferred in may not be what the instructor intended or anticipated.) The inclusion of transfer in allows these researchers to expand transfer to include *preparation for future learning* (PFL), given the appropriate environment.

One consequence of these more recent transfer definitions is that a researcher may not observe transfer unless the research task or assessment is explicitly designed to elicit it. Wagner argues that transfer in pieces will be observed only in teaching experiments or repeated interviews with the same student in which different tasks are provided that are situated in different contexts connected by the same concept: individual tasks do not demonstrate a student's ability to learn incrementally across contexts. Similarly, Bransford and colleagues argue that only specific types of research tasks—those that elicit learning during the activity (i.e., transfer in) and probe for interpretive use of prior knowledge—have the possibility of demonstrating preparation for future learning. Bransford and Schwartz refer to the tasks and skills seen in traditional transfer experiments as *sequestered problem solving* (SPS). In both of these frameworks, transfer experiments must involve some guiding of, teaching to, or prompting for the student. In other words, in order to make substantial claims about transfer, experiments and protocols would have to be designed that specifically target transfer mechanisms. The transfer-in-pieces model suggests that our students had not conducted enough targeted analysis of Taylor series in different contexts prior to our data collection in order to coordinate their knowledge in each context and learn from each new context. Facets of their knowledge were coordinated to some extent, e.g., their ability to graphically interpret the coefficients of a Taylor series, but other facets, such as how to generate a Taylor series in a specific situation and how to use it to make an approximation, were not sufficiently coordinated. This lack of coordination prevented across-context recognition of the salient features of the Taylor series, thus preventing abstraction, which may be what is needed to generate and use a Taylor series in a novel context.

The model of transfer as it relates to PFL suggests two interpretations of our results. Our data indicate that students could, for the most part, execute a Taylor expansion when asked to do so; this is evidence of efficiency with this fairly algorithmic task. The fact that it was not universal is evidence that these students are not yet "routine experts" [91] in the execution of Taylor expansions, especially in the novel context of entropy as a function of energy. (Recall the student Joel, who accurately derived the Boltzmann factor in a teaching interview, but was unable to describe the physical meaning of the terms therein. Joel could be granted routine expert status in this case, given his rapid and accurate recall of the derivation.) Additionally, none of the interviewed students could spontaneously recognize the need for a Taylor series expansion or approximation in the context of the Boltzmann factor derivation. This suggests that innovation, manifested as the ability to interpret the scenario as one needing this expansion, is lacking on the part of the students.

The difference between these interpretations has to do with the tasks used to collect the data. The more formal assessment tasks on the use of a Taylor series in statistical mechanics—the written pretests and clinical interviews—would be considered SPS tasks; thus one might argue that these kinds of assessments automatically fail to see transfer due to their limited expectations. On the other hand, the more authentic settings of the pretutorial homework, the tutorial activity during class, and the teaching interviews should have been sufficiently genuine to exhibit learning and transfer, provided students were so prepared. If only SPS tasks were analyzed, we would have difficulty concluding anything about the presence or absence of transfer. However, because our methods included PFL tasks, we are able to conclude that in this case the students did not exhibit transfer: prior knowledge did not prepare students for learning in this context. (Again, Joel serves as an example of being able to recall prior knowledge without any accompanying interpretive skills to use with that knowledge.) The difficulties students have with recognizing a need for invoking the Taylor series when the situation calls for it suggest a lack of transfer of this knowledge from other situations (e.g., kinematics and electrostatics), to the extent that the tasks used here could have elicited this kind of transfer.

This brief analysis demonstrates that our results can be viewed through these transfer lenses and yield reasonable interpretations consistent with the existing models. However, as mentioned above, explicit attention to transfer in these frameworks requires specific experimental conditions that we were not concerned with in our study (of student understanding of the derivation of the Boltzmann factor). Further research to collect different data and/or a modified analysis of our existing data are needed to provide a more thorough interpretation.

VII. CONCLUSIONS AND IMPLICATIONS FOR FUTURE WORK

Our studies on student understanding of a Taylor series expansion in a derivation of the Boltzmann factor have provided mixed results. Data from the Taylor series pretest indicate that many students were able to interpret Taylor series expansions of a function given the graph of that function. Results from interviews and classroom observations, however, indicate that students struggle to generate a Taylor series expansion in a physical context (e.g., with entropy and energy). Once provided with a generic Taylor series using physical quantities, most students were able to apply it to a specific situation, but this was usually not a trivial task for them, as seen during our teaching interviews and classroom observations.

These results underscore the need for preparation such as that provided by the pretutorial homework assignment in which students are asked to generate the Taylor expansion in Eq. (5). We have found this pretutorial homework strategy to be worthwhile, even necessary, for implementing tutorials in upper-division thermal physics courses. This marks a distinct difference from typical tutorial implementation in introductory courses, as far more prerequisite knowledge is both required and assumed at the upper division, including a robust understanding of concepts in both physics and mathematics.

Results from clinical interviews on student understanding of the applicability of Taylor series expansions show that many students recognized that the Taylor series is a relevant mathematical tool in various areas of physics, but they often lacked a sense of when and how its use is appropriate. Students also did not have rigorous criteria for determining how many terms should be kept (except when one of the derivatives is a constant, resulting in all higher derivatives being identically zero).

In our opinion, the application of a Taylor series expansion in a specific physical context, especially one involving entropy and energy, is quite a sophisticated and complex process. It involves not only recall of the mathematical expansion, but an understanding of the mathematical meaning of each term, a physical interpretation of how each of those terms relates to the given physical situation, and judgment of the appropriate conditions to apply for series termination. This also assumes that a student recognizes that a Taylor series expansion could make the problem tractable. When abstract physical quantities known to be conceptually difficult are involved, the cognitive load quickly escalates.

We suggest several additional research questions based on our results: Under what conditions might students choose (or are they able) to appropriately and productively use a Taylor series expansion without instructor intervention? What aspects of a physical context should be highlighted to encourage its use? Are some physical quantities easier for students to use in a Taylor series expansion? Alternatively, are entropy and energy too abstract for our students to use with sophisticated mathematical tools like Taylor series expansion? Of particular interest would be an examination of expert physicists' spontaneous use of the Taylor series. Spontaneous use of prior knowledge in a flexible manner would be consistent with the PFL model for transfer. Capturing this expert "transfer in" in the context of Taylor series expansions would provide models for instructional resource design to enhance student understanding of physics and useful mathematical tools within many different courses typically taught in the undergraduate—and graduate—curriculum. This would provide an excellent stepping stone for undergraduate physics majors on their way to expertise.

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APPENDIX A: TEACHING INTERVIEW ACTIVITY

The following activity was used during the teaching interviews conducted during year 1 of the study.

Systems and Reservoirs

Consider a container of an ideal gas isolated from its surroundings (Fig. 3). The container is divided into two sections: a relatively small section ($\mathcal{C}$) that will be our system of interest and a relatively large section ($\mathcal{R}$). The two sections are in thermal equilibrium and have uniform spatial density, and the combined energy is equal to E_{tot} (i.e., $E_{\mathcal{C}} + E_{\mathcal{R}} = E_{tot}$). Since $\mathcal{R}$ is so much larger than $\mathcal{C}$ we will treat $\mathcal{R}$ as a thermal reservoir. We know from chapter 4 [of Ref. [51]] and the density of states tutorial that the energy of a system in thermal equilibrium may fluctuate around an average value ($E_{\mathcal{C}} = E_{Ave} \pm \delta E$). We also know that the multiplicity of an ideal gas is related to the volume of the gas, its internal energy, and the number of particles ($\omega \propto V^N E^{3N/2}$). Therefore we may conclude that the multiplicity of $\mathcal{R}$ will be very much larger than the multiplicity of $\mathcal{C}$ (i.e., $\omega_{\mathcal{R}} \gg \omega_{\mathcal{C}}$). As such, we will make the approximation that $\omega_{\mathcal{R}}/\omega_{\mathcal{C}} \approx \omega_{\mathcal{R}}$ which leads to $\omega_{\mathcal{C}} \approx 1$. For the remainder of our discussion we will investigate a model in which $\omega_{\mathcal{C}} = 1$ and the fluctuations in $E_{\mathcal{C}}$ will yield a handful of discrete values ($E_{Ave} \pm \delta E = E_j = E_1, E_2, E_3, \ldots$). The table below shows a scenario in which there exist only 5 possible values for $E_{\mathcal{C}}$, each with corresponding values for $E_{\mathcal{R}}$, $\omega_{\mathcal{C}}$, and $\omega_{\mathcal{R}}$.

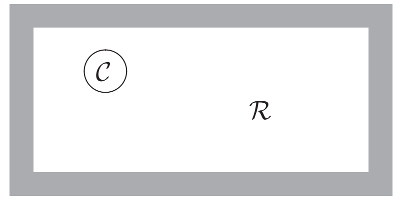

FIG. 3. Figure provided for Systems and Reservoirs Activity.

$E_{\mathcal{C}}$	$\omega_{\mathcal{C}}$	$E_{\mathcal{R}}$	$\omega_{\mathcal{R}}$
E_1	1	$E_{tot} - E_1$	3×10^{18}
E_2	1	$E_{tot} - E_2$	5×10^{19}
E_3	1	$E_{tot} - E_3$	4×10^{17}
E_4	1	$E_{tot} - E_4$	1×10^{20}
E_5	1	$E_{tot} - E_5$	7×10^{18}

(A) What is the total number of *microstates* for the entire container (system + reservoir) in our scenario?
(B) Are any of the *microstates* more probable than any other?
(C) Using your answer to part B, which of the above *macrostates* is most probable? Why? Which *macrostate* is least probable?
(D) Give a general expression for the probability, $P(E_j)$, of $E_{\mathcal{C}} = E_j$.

Energy, Entropy, and Probability

You've now determined that the probability of the system $E_{\mathcal{C}}$ having energy E_j, $P(E_j)$, is *proportional* to the multiplicity of the reservoir for that state, labeled $\omega_{\mathcal{R}j}$. (Compare this to the probability you've found previously for getting M heads from flipping N coins.) But what if we don't explicitly know $\omega_{\mathcal{R}j}$, as will often be the case in real systems? In this case, we need an expression for $\omega_{\mathcal{R}j}$ that depends on properties of $E_{\mathcal{C}}$ (i.e., $\omega_{\mathcal{R}j} = \omega_{\mathcal{R}j}(E_j, T_{\mathcal{C}}, V_{\mathcal{C}}, \ldots)$).

(A) Is state j a macrostate or a microstate? How do you know?
(B) Write an expression for the entropy of the reservoir ($S_{\mathcal{R}j}$) in terms of $\omega_{\mathcal{R}j}$.
(C) Now use Taylor series expansion and the fact that entropy is a function of energy ($S_{\mathcal{R}j} = S_{\mathcal{R}j}(E_{\mathcal{R}j})$) to write an approximation for $S_{\mathcal{R}j}$ as a linear function of E_j. [Note: If needed, students were given the related Taylor series expansion of $S(E)$ centered at $E = E_0$ (shown in Eq. 5) as a reference for generating the desired expression (Eq. 4).]
What is the *physical* interpretation of the first term in the Taylor expansion? Does this fit with what you know about Taylor series? Rename the first term to reflect this interpretation.
What is the *physical* interpretation of the partial derivative in the second term? Consider the differential form of the first law of thermodynamics.
(D) Equate your two expressions for $S_{\mathcal{R}j}$ from parts B and C to get an expression for $\omega_{\mathcal{R}j}$ in terms of the other variables and constants.
Which of these quantities will change with different values of j?
(E) Since $P(E_j) \propto \omega_{\mathcal{R}j}$ we can group any constant coefficients together. Write an expression for $P(E_j)$ as a function of E_j eliminating any constant terms and dividing by a normalizing term Z. (Remember, a function of a constant is a constant.)
(F) Determine an expression for Z and rewrite your expression for $P(E_j)$. Consider the constraint on the sum over all probabilities $P(E_j)$.
(G) Is your new expression for Z a constant? (i.e., does it depend on the state of the system?) How does your expression for Z compare with the normalizing factor for the binomial distribution (2^N)?

The normalizing factor for the probability is known as the canonical partition function. The symbol Z comes from the German *Zustandsumme* meaning "sum over states."

APPENDIX B: CLINICAL INTERVIEW PROTOCOL

The following interview protocol was used during the clinical interviews conducted during year 2 of the study.

(1) What do you know about the Taylor series? That is, when I say "Taylor series," what comes to mind?
(2) How do you know how many terms to write?
(3) Why do we even care about the Taylor series? Is it applicable in physics?
  (i) If so, when?
  (ii) How does it relate to perturbation theory?

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