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Partitions of finite frames

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PARTITIONS OF FINITE FRAMES

by

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Dedications

I dedicate this thesis to my sister Victoria and my parents Maria and Victor.

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Abstract

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An open question stated by Marcus, Spielman, and Srivastava [10] asks "whether one can design an efficient algorithm to find the partitions guaranteed by Corollary 1.5." This corollary states that given a set of vectors in $\mathbb{C}$ whose outer products sum to the identity there exists a partition of these vectors such that norms of the outer-product sums of each subset satisfy an inequality bound. Here particular types of vector sets called finite frames are analyzed and constructed to satisfy the inequality described in Corollary 1.5. In this thesis, rigorous proofs and formula-tions of outer-product norms are utilized to find these partitions and to identify constraints on the finite frames in order to satisfy Corollary 1.5.

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Chapter 1

Introduction

The Kadison-Singer Conjecture and Frames

In 1959 the Kadison-Singer problem was posed by Richard Kadison and Isadore Singer after exploring Dirac's research in quantum mechanics in the 1940s. The problem is stated below [4],

Kadison-Singer Problem (KS)

Does every pure state on the (abelian) von Neumann algebra $\mathbb{D}$ of bounded diagonal operators on the Hilbert space l_2 have a unique extension to a (pure) state on $B(l_2)$, the von Neumann algebra of all bounded linear operators on l_2?

Let us give a run down of the terminology stated by the Kadison-Singer problem (KS). A *state* is a linear functional that maps a vector in a Hilbert space $\mathcal{H}$, which could be in $\mathbb{R}$ or $\mathbb{C}$, to its scalars in that given Hilbert space $\mathcal{H}$. As an example, suppose you have a vector $x \in \mathbb{R}^n$, and each component $x_i \in \mathbb{R}$; therefore, our domain is Hilbert space the $\mathbb{R}^n$. A linear functional acting on this vector could be described as $f(x) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$, where $\alpha_1, \alpha_2, \ldots$ are arbitrary real numbers. Notice that this is the mapping defined by $f : \mathbb{R}^n \to \mathbb{R}$ where $\{x_k\}_{k=1}^n \mapsto c$, a scalar. Next one can think of the KS problem as utilizing matrices or more generally "algebras." With the KS problem we are examining *diagonal* operators as our linear functionals, which are mappings between spaces with respect to diagonal elements of matrices. As it turns out there are many *states*, linear functionals, and all together they form a set of states that act on these algebras. The *pure states* are the extremal elements of this set of states. The KS problem probes whether the said diagonal operators on our Hilbert space $\mathcal{H}$ all have corresponding extensions to linear operators in our same given space $\mathcal{H}$. An *extension* on a state is a linear functional that "reduces" to another linear functional acting on the same space. For example, suppose you have a function $g(a, b, c, d) = f(a, d)\ \forall a, b, c, d \in \mathbb{R}$, then g is the extension of f over the given Hilbert space $\mathbb{R}$. This is so because even though g contains more input values than f it still reduces to the same output values that f encompasses. Therefore, we have extended the domain of f to the domain of g, but the two have the same range of values.

Originally, Kadison conjectured the solution to the KS problem was negative; however, in 2013 Marcus, Spielman, and Srivastava provided a positive solution [10] to the KS problem. Their solution was found by reformulating the KS problem into equivalent forms; for instance, Weaver demonstrated in [15] that the KS problem can be thought of as a combinatorial problem which also addresses vectors in $\mathbb{C}^n$ and finite matrices in complex Hilbert spaces. The KS problem is also equivalent to the paving conjecture proposed by Anderson [1] which relates linear operators to diagonal operators in a complex Hilbert space. In [10], the authors utilized these equivalent forms of the KS problem and analysis of the largest roots to *mixed characteristic polynomials* of a set of matrices to prove the paving conjecture and Weaver's conjecture; hence, this resulted in a positive solution of the KS problem.

In the proof that Marcus, Spielman, and Srivastava provided, they state Corollary 1.5, which describes an upper bound to the norms of special matrices that are formed from special partitionings (groupings) of vectors contained in a collection. One of the open questions posed by the authors at the end of their paper [10] is whether one can formulate an *efficient* algorithm that partitions a given collection of vectors such that it satisfies Corollary 1.5. This thesis explores a special type of vector collection known as *frames*. We show that these frames meet the hypothesis of the corollary and explore ways of finding partitions of these frames to satisfy the conclusion of Corollary 1.5.

The construction of frames has its origins in harmonic and functional analysis, operator theory, linear algebra, and matrix theory. They were a result of the limited capabilities of regular bases being utilized in signal processing and sampling theory. Originally, transmission signals were decomposed and expanded using the Fourier transform, but this left out or hid some important components contained in the signal. Gabor rectified this by embedding more and redundant signal information into the transmission [9]. This extra information is unique to frames and Gabor's efforts marked the early steps in the construction of frames. Unlike orthonormal bases which have a limited number of vectors and no redundancy, a frame can carry more information about the original signal being transmitted. The formal definition of a frame was formulated by Duffin and Schaeffer in 1952 [12], and they utilized to compute the coefficients of linear combinations of vectors that were linearly dependent in a given Hilbert space. A frame in essence can be thought of as a relaxation of a basis, such that linear *independence* is not a requirement. Thus we can have repeated elements in our frame, but the set of vectors in the frame must still *span* the Hilbert space that we are working in. In addition to these qualities, a frame has what are known as *frame bounds*, which restrict the maximum and minimum sizes of sums of inner products of a given vector with our frame vectors. As a result of having frames, the challenges of reconstructing a signal with minimal loss of information was remedied. This has to do with the fact that frames are not as restrictive in their features compared to regular bases.

The first part of this chapter was dedicated to giving a brief history of the Kadison-Singer problem and its implications to finding partitions of finite frames. In the rest of this chapter we will delve into the preliminary theorems involving the KS problem and how finite frames can be associated with Corollary 1.5. The second chapter explores a class of finite frames that are equinormal and equiangular. Moreover, we shall prove various ways of partitioning the frame given certain frame characteristics. In the third chapter we will examine Grassmannian frames and show that the size of the Grassmannian frame affects whether the frame subsets will meet the bound described by Corollary 1.5. Throughout this thesis we will designate partitions and the corresponding subsets that meet the corollary bound as being *valid*.

Our research has yielded the following results and questions that still need to be addressed. Some results:

- We have found ways to partition equinorm-equiangular frames using algebraic and numerical analysis.

- We have shown numerically that for *small* order Grassmannian frames any partition will be valid.

Some open problems that are still being analyzed are:

- How do we find the eigenvalues for Hermitian matrices generated by outer products in $\mathbb{C}^{n\times n}$ where n is large?
- How does the choice of the frame vectors affect the corresponding norms of the outer products, particularly for higher-order Grassmannian frames?
- Once we obtain formulas for the norms of partial frames, then can we find *efficient* algorithms to partition these frames?

Preliminaries

Let us begin by stating Corollary 1.5 from [10]:

Theorem 1.1 (MSS Corollary 1.5). *Let r be a positive integer and let* $u_1, ..., u_m \in \mathbb{C}^d$ *be vectors such that*

$$\sum_{i=1}^{m} u_i u_i^* = I \tag{1.1}$$

and $\|u_i\|^2 \leq \delta$ *for all i. Then there exists a partition* $\{S_1, ..., S_r\}$ *of* $[m] = \{1 \ldots m\}$ *such that*

$$\left\| \sum_{i\in S_j} u_i u_i^* \right\| \leq \left(\frac{1}{\sqrt{r}} + \sqrt{\delta} \right)^2 \tag{1.2}$$

for $j = 1, ..., r$.

As a side note, the calculation uu^* in $\mathbb{C}^{d\times d}$ and uu^T in $\mathbb{R}^{d\times d}$ is known as the *outer product* or *tensor product* of two vectors and is used extensively in this thesis. In essence, given a set of d dimensional vectors in the complex domain with square magnitudes less than or equal to a given constant δ, we can always group the vectors into subsets such that the *norm* of the outer-product sum never exceeds a certain value (upper bound). The norm that we will calculate is defined by Meyer [11]:

Definition 1.1 (Induced Matrix Norms)**.** *A vector norm that is defined on $\mathbb{C}^n$, induces a matrix norm on $\mathbb{C}^{n\times n}$ by setting*

$$\|\mathbf{A}\| = \max_{\|x\|=1} \|\mathbf{A}x\|,$$

where $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $x \in \mathbb{C}^{n\times 1}$.

As a result of this definition we need to find the spectral values, the λ's, of our matrices by calculating the solutions to $\det(\mathbf{A} - \lambda I) = 0$. The maximum of these spectral values is our norm; therefore

$$\|\mathbf{A}\| = \max\{\lambda_1, \lambda_2, \ldots\}.$$

What makes the Corollary interesting is that the upper bound only depends on the number of "subsets" and the constant δ. The following conjecture [10] (KS_2) precedes Corollary 1.5:

Theorem 1.2 (Conjecture 1.2 (KS_2))**.** *There exist universal constants $\eta \geq 2$ and $\theta > 0$ so that the following holds. Let $w_1, ..., w_m \in \mathbb{C}^d$ satisfy $\|w_i\| \leq 1$ for all i and suppose*

$$\sum_{i=1}^{m} |\langle u, w_i \rangle|^2 = \eta$$

for every unit vector $u \in \mathbb{C}^d$. Then there exists a partition S_1, S_2 of $[m]$ so that

$$\sum_{i \in S_j} |\langle u, w_i \rangle|^2 \leq \eta - \theta. \tag{1.3}$$

Therefore, the Conjecture 1.2 is a special case of Corollary 1.5. Here we are finding a "2-subset partition" of a set of vectors in $\mathbb{C}^d$, this means $[m] = S_1 \cup S_2$. Notice that the conclusion of Conjecture 1.2 is reminiscent of a what is known as a *finite frame*. Below is the definition as described in [5, 6, 8].

Definition 1.2. *A frame for a Hilbert space $\mathcal{H}$ is a sequence of vectors $\{f_i\} \subset \mathcal{H}$ for which there exist constants $0 < A \leq B < \infty$ such that, for every $x \in \mathcal{H}$,*

$$A\|x\|^2 \leq \sum_i |\langle x, f_i \rangle|^2 \leq B\|x\|^2, \tag{1.4}$$

where the values A and B are the frame bounds.

Notice that if we express

$$u = \frac{x}{\|x\|},$$

then (1.3) can be rewritten as follows:

$$\sum_{i \in S_j} \frac{|\langle x, w_i \rangle|^2}{\|x\|^2} \leq \eta - \theta \Longrightarrow \sum_{i \in S_j} |\langle x, w_i \rangle|^2 \leq (\eta - \theta)\|x\|^2. \tag{1.5}$$

By comparing (1.4) and (1.5) we have that the upper frame bound is given by $B = \eta - \theta$ and the frame vectors become $f_i = w_i$. A natural question that arises is whether Corollary 1.5 holds for frames given that they are applicable to Conjecture 1.2. Let us examine (1.1) more closely by taking the norm of the sum of outer products and utilizing the Triangle Inequality for matrices to get

$$\left\|\sum_{i=1}^{m} u_i u_i^*\right\| \leq \sum_{i=1}^{m} \|u_i u_i^*\|.$$

But recall from the hypothesis of Corollary 1.5 that $\|u_i\|^2 \leq \delta$. We combine this with the identity $\|uu^*\| = \|u\|^2$ (A.4) to get

$$\begin{aligned}\left\|\sum_{i=1}^{m} u_i u_i^*\right\| &\leq \sum_{i=1}^{m} \|u_i u_i^*\| \\ &= \sum_{i=1}^{m} \|u_i\|^2 \\ &\leq m\delta. \end{aligned} \tag{1.6}$$

We then multiply (1.6) by $\|x\|^2$ to get

$$\begin{aligned}\sum_{i=1}^{m} \|u_i\|^2 \|x\|^2 &= \sum_{i=1}^{m} (\|u_i\|\|x\|)^2 \\ &\leq m\delta\|x\|^2. \end{aligned} \tag{1.7}$$

By the CBS inequality we get

$$\sum_{i=1}^{m} |\langle x, u_i\rangle|^2 \leq \sum_{i=1}^{m} \|x\|^2 \|u_i\|^2 \leq m\delta\|x\|^2. \tag{1.8}$$

So clearly based on the hypothesis of Corollary 1.5 we can achieve an upper frame bound for the vectors $u_1, ..., u_m$. In fact, we can apply the same steps used to find (1.8) on the subsets of the partitions described by the conclusion of Corollary

1.5 to get

$$\sum_{i \in S_j} |\langle x, u_i \rangle|^2 \leq |S_j| \delta \|x\|^2, \tag{1.9}$$

where $m = |S_j|$ is the cardinality of the set S_j. It appears that utilizing finite frames will be appropriate in determining an algorithm for partitioning a frame because (1.9) is analogous to the definition of a frame with upper bound $B = |S_j|\delta$.

What are Finite Frames?

To describe what a frame is we recall the notion of a vector basis. As described by Meyer [11], a basis is *a linearly independent spanning set of vectors for a vector space $\mathcal{V}$*. In other words, a basis is a collection of vectors that can be used to generate all the other vectors through linear combinations. Moreover, the vectors in this collection are *independent* of each other. This idea of independence means that one vector in our collection cannot be formed from any combination of the other vectors in the collection. This definition lends it self to some nice properties involving bases, but at the same time makes bases too restrictive when applying them to signal processing and reconstruction problems. We shall walk through two nice examples of bases:

- The standard unit vectors in $\mathbb{R}^n$
- The orthonormal basis in a space $\mathcal{V}$

Standard unit vectors. As an example, the standard unit vectors in $\mathbb{R}^2$ are

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Notice that any two values $x, y \in \mathbb{R}$ can be utilized as coefficients for a linear combination of e_1 and e_2. If one would like to form a new vector v with components x and y, the we simply calculate

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = xe_1 + ye_2.$$

Suppose we let $x = 1$ and $y = 2$, then our example vector v is

$$v = 1 \cdot e_1 + 2 \cdot e_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Since we can choose any $x, y \in \mathbb{R}$ to create a vector $v \in \mathbb{R}^2$, then this means that our two vectors e_1 and e_2 span the real plane $\mathbb{R}^2$. Also note that e_1 and e_2 are independent of each other. Meaning that the solution to the equation

$$c_1e_1 + c_2e_2 = 0$$

is $\{c_1, c_2\} = \{0, 0\}$; therefore, it is impossible for one vector to be expressed in terms of another vector without triviality. Hence, the set $\mathcal{S} = \{e_1, e_2\}$ is a basis in $\mathbb{R}^2$. In fact, we can generalize this to the set of standard unit vectors $\mathcal{S} = \{e_1, e_2, ..., e_n\}$ $\forall e_i \in \mathbb{R}^n$ and this also forms a basis in $\mathbb{R}^n$. Notice that these vectors do not repeat in $\mathcal{S}$, nor do the vectors *depend* on each other.

Orthonormal bases. Another familiar type of basis is the *orthonormal basis* for a vector space $\mathcal{V}$ with an inner product $\langle \cdot, \cdot \rangle$. The definition is similar to that of a general basis except we have the following two restrictions [11]:

- Each vector has unit length (hence "normal").
- The set of vectors form an orthogonal set.

In essence, suppose you have a set $\mathcal{B} = \{u_1, u_2, ..., u_n\}$ where each u_i is from an n-dimensional vector space $\mathcal{V}$. Then we say that $\mathcal{B}$ is an orthonormal basis if for all $u_i, u_j \in \mathcal{B}$

$$\langle u_i, u_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} .$$

Keep in mind you may loosely think of "orthogonal" to mean perpendicular. Since all the vectors are orthogonal to each other, then no one vector in $\mathcal{B}$ can be expressed as the linear combination of the other remaining vectors. Therefore, the only solution set to the coefficients satisfying the equation

$$\sum_j c_j u_j = 0$$

is $\{c_1, c_2, ..., c_n\} = \{0, 0, ..., 0\}$. This means the vectors in $\mathcal{B}$ are linearly independent and form an orthonormal basis for $\mathcal{V}$.

Notice that the number of vectors described by the definition of a basis is limited to the number of dimensions of the vector space $\mathcal{V}$. For instance, the standard orthonormal basis in $\mathbb{R}^n$ will have exactly n vectors.

To summarize, a *frame* can be thought of as a "relaxed" form of a basis. One of the key components is that a frame may contain repeated vectors and/or vectors that are *not* linearly independent. The second is that it still spans the vector space. For example, let

$$\mathcal{F} = \{e_1, e_2, -e_1, 2e_2\}$$

which contains linear combinations of the standard basis in $\mathbb{R}^2$ and still spans $\mathbb{R}^2$. If we were to utilize these previously discussed bases to transmit a signal, the vectors would encounter natural perturbations during transmission. Some of these changes can be attributed to noise, erasures, or other natural phenomena that mutate the receiving signal [5]. At any rate with standard unit vectors or orthonormal bases we have a limited number of vectors to choose from and there is no guarantee that all of the extracted signal using these vectors will survive the actual transmission process. Frames remedy this problem by providing redundancy to a transmitted signal.

Chapter 2

Partitions of Frames

Equinorm-Equiangular Frames

Equinorm and equiangular frames in $\mathbb{R}^2$ can be thought of as vectors that have the same angle between them. We will define this frame in the following way:

$$\mathcal{F} = \{f_j\}_{j=0}^{m-1} = \sqrt{\frac{2}{m}} \begin{bmatrix} \cos(\frac{2\pi j}{m}) \\ \sin(\frac{2\pi j}{m}) \end{bmatrix}_{j=0}^{m-1}. \tag{2.1}$$

Notice that the norm for every vector f_j equals $\sqrt{\frac{2}{m}}$ and the angle between two consecutive vectors is always $\frac{2\pi}{m}$ radians. As an example if we let $m = 3$, then our frame would be

$$\mathcal{F} = \{f_0, f_1, f_2\} = \left\{ \begin{bmatrix} \sqrt{\frac{2}{3}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right\}.$$

Figure 1 is the Mercedes-Benz Frame placed in the coordinate plane, called so because it is similar to the logo of the auto manufacturer Mercedes-Benz.

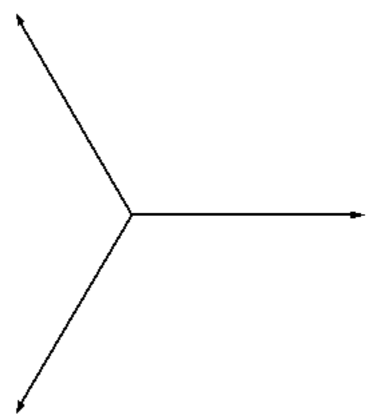


Figure 1. Mercedes-Benz Frame, $m = 3$

Let us find the frame bounds of (2.1). To do this we will examine the summation found in (1.4). Let $x = \{\alpha, \beta\}$ where $\alpha, \beta \in \mathbb{R}$. Then

$$\sum_{j=0}^{m-1} |\langle x, f_j \rangle|^2 = \frac{2}{m} \sum_{j=0}^{m-1} \left| \alpha \cos\left(\frac{2\pi j}{m}\right) + \beta \sin\left(\frac{2\pi j}{m}\right) \right|^2$$

$$= \frac{2}{m} \Bigg(\alpha^2 \sum_{j=0}^{m-1} \cos^2\left(\frac{2\pi j}{m}\right)$$

$$+ 2\alpha\beta \sum_{j=0}^{m-1} \cos\left(\frac{2\pi j}{m}\right) \sin\left(\frac{2\pi j}{m}\right) + \beta^2 \sum_{j=0}^{m-1} \sin^2\left(\frac{2\pi j}{m}\right) \Bigg). \tag{2.2}$$

If we apply the trigonometric identities for double angles, (B.3), and (B.4) on the three summations in (2.2), then we have

$$\begin{aligned} \sum_{j=0}^{m-1} \cos^2\left(\frac{2\pi j}{m}\right) &= \frac{1}{2} \sum_{j=0}^{m-1} \left(1 + \cos\left(\frac{4\pi j}{m}\right)\right) \\ &= \frac{m}{2} + \frac{1}{2} \sum_{j=0}^{m-1} \cos\left(\frac{4\pi j}{m}\right) \\ &= \frac{m}{2}, \end{aligned} \tag{2.3}$$

$$\sum_{j=0}^{m-1} \cos\left(\frac{2\pi j}{m}\right) \sin\left(\frac{2\pi j}{m}\right) = \frac{1}{2} \sum_{j=0}^{m-1} \sin\left(\frac{4\pi j}{m}\right) = 0, \tag{2.4}$$

and

$$\sum_{j=0}^{m-1} \sin^2\left(\frac{2\pi j}{m}\right) = \frac{1}{2}\sum_{j=0}^{m-1}\left(1 - \cos\left(\frac{4\pi j}{m}\right)\right)$$
$$= \frac{m}{2} - \frac{1}{2}\sum_{j=0}^{m-1}\cos\left(\frac{4\pi j}{m}\right)$$
$$= \frac{m}{2}. \tag{2.5}$$

Now substitute (2.3), (2.4), and (2.5) back into (2.2) to get

$$\sum_{j=0}^{m-1} |\langle x, f_j\rangle|^2 = \frac{2}{m}\left(\alpha^2 \cdot \frac{m}{2} + 2\alpha\beta \cdot 0 + \beta^2 \cdot \frac{m}{2}\right)$$
$$= \alpha^2 + \beta^2$$
$$= \|x\|^2. \tag{2.6}$$

Interestingly, if we compare the result (2.6) to the frame definition (1.4) the bounds of the frame are equal, namely $A = B = 1$. When $A = B$ the frame is known as a *tight* frame. If both frame bounds equal 1, then it is called a *Parseval* frame.

We now show that the frame described by (2.1) satisfies the hypothesis of MSS

Corollary 1.5:

$$\sum_{j=0}^{m-1} f_j f_j^T = \frac{2}{m} \sum_{j=0}^{m-1} \begin{bmatrix} \cos^2\left(\frac{2\pi j}{m}\right) & \cos\left(\frac{2\pi j}{m}\right)\sin\left(\frac{2\pi j}{m}\right) \\ \cos\left(\frac{2\pi j}{m}\right)\sin\left(\frac{2\pi j}{m}\right) & \sin^2\left(\frac{2\pi j}{m}\right) \end{bmatrix}$$

$$= \frac{2}{m} \cdot \frac{1}{2} \sum_{j=0}^{m-1} \begin{bmatrix} 1+\cos\left(\frac{4\pi j}{m}\right) & \sin\left(\frac{4\pi j}{m}\right) \\ \sin\left(\frac{4\pi j}{m}\right) & 1-\cos\left(\frac{4\pi j}{m}\right) \end{bmatrix}$$

$$= \frac{1}{m} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$= I.$$

It suffices t o n ow fi nd pa rtitions fo r th ese eq uinorm-equiangular fr ames guaranteed by MSS Corollary 1.5, which we investigate next.

Partitions of Equinorm-Equiangular Frames

In this section we will examine the partitions of equinorm-equiangular frames. Our analysis will involve partitioning the frame $\mathcal{F}$ into two subsets, S_1 and S_2, where $|S_1| + |S_2| = m$. We then compare the norm of the sum of the outer prod-ucts to the bound in MSS Corollary 1.5 [10]. First we shall establish a relationship between the vectors corresponding to the subsets S_1 and S_2 to the discriminant cal-culated from the norm of a matrix. Second we will examine algorithms for three special cases of partitions:

- $|S_1| = |S_2|$

- $|S_1| \neq |S_2|$

- $|S_1| = |S_2| + 1$

The discriminant problem. We will now begin examining the norm of the matrix formed from the sum of outer products of vectors corresponding to a subset S_k. Recall that $[m] = S_1 \cup S_2$. Our goal in this section is to show the existence of a relationship between the vectors in a subset to the discriminant corresponding to the norm of the outer product sum.

- First we will derive the sum of outer products when $|S_k| = 2 \Rightarrow S_k$

- then we will extend it to the case when $|S_1| > 2$.

Outer Product Sum for 2 Vectors in a Subset. Suppose we let $S_1 = \{a_1, a_2\}$ where $a_1, a_2 \in [m]$. If we utilize the trigonometric double angle identities and compute the sum of the outer products of frame vectors corresponding to the elements in our subset S_1 we have

$$
\begin{aligned}
\mathbf{F}_{S_1} &= \sum_{j\in S_1} f_j f_j^T \\
&= \frac{2}{m}\sum_{j\in S_1}\begin{bmatrix}\cos\left(\frac{2\pi j}{m}\right)\\ \sin\left(\frac{2\pi j}{m}\right)\end{bmatrix}\begin{bmatrix}\cos\left(\frac{2\pi j}{m}\right) & \sin\left(\frac{2\pi j}{m}\right)\end{bmatrix} \\
&= \frac{1}{m}\sum_{j\in S_1}\begin{bmatrix} 2\cos^2\left(\frac{2\pi j}{m}\right) & 2\cos\left(\frac{2\pi j}{m}\right)\sin\left(\frac{2\pi j}{m}\right)\\ 2\cos\left(\frac{2\pi j}{m}\right)\sin\left(\frac{2\pi j}{m}\right) & 2\sin^2\left(\frac{2\pi j}{m}\right)\end{bmatrix} \\
&= \frac{1}{m}\begin{bmatrix} 2+\cos\left(\frac{4\pi a_1}{m}\right)+\cos\left(\frac{4\pi a_2}{m}\right) & \sin\left(\frac{4\pi a_1}{m}\right)+\sin\left(\frac{4\pi a_2}{m}\right)\\ \sin\left(\frac{4\pi a_1}{m}\right)+\sin\left(\frac{4\pi a_2}{m}\right) & 2-\cos\left(\frac{4\pi a_1}{m}\right)-\cos\left(\frac{4\pi a_2}{m}\right)\end{bmatrix}.
\end{aligned}
$$

Notice that the matrix $\|\mathbf{F}_{S_1}\|$ is Hermitian.

Outer Product Sum for n Vectors in a Subset. Next we can generalize this to the case where $S_1 = \{a_1, a_2, ..., a_n\}$ and $|S_1| = n$ is any positive number less than m. Fix $|S_2| = p$. We have

$$
\mathbf{F}_{S_1} = \frac{1}{m}\begin{bmatrix} n+\sum_{j\in S_1}\cos\left(\frac{4\pi j}{m}\right) & \sum_{j\in S_1}\sin\left(\frac{4\pi j}{m}\right)\\ \sum_{j\in S_1}\sin\left(\frac{4\pi j}{m}\right) & n-\sum_{j\in S_1}\cos\left(\frac{4\pi j}{m}\right)\end{bmatrix} \tag{2.7}
$$

and

$$\mathbf{F}_{S_2} = \frac{1}{m}\begin{bmatrix} p + \sum_{j\in S_2}\cos\left(\frac{4\pi j}{m}\right) & \sum_{j\in S_2}\sin\left(\frac{4\pi j}{m}\right) \\ \sum_{j\in S_2}\sin\left(\frac{4\pi j}{m}\right) & p - \sum_{j\in S_2}\cos\left(\frac{4\pi j}{m}\right) \end{bmatrix}. \tag{2.8}$$

Therefore, (2.7) and (2.8) give the generalized outer product sums of the frame vectors extracted by the subsets S_1 and S_2.

Our goal is to find the norm (2.7) and compare it to the bound described by (1.2). Since we are dealing with two subsets of $\mathcal{F}$ we have $r = 2$ and the largest modulus of each frame vector f_j is $\sqrt{\frac{2}{m}}$ so that $\delta = \frac{2}{m}$. If suffices to determine S_1 so that (1.2) holds:

$$\|\mathbf{F}_{S_1}\| \leq \left(\frac{1}{\sqrt{2}} + \sqrt{\frac{2}{m}}\right)^2. \tag{2.9}$$

Let us determine the norm of the matrix $\mathbf{F_{S_1}}$. To simplify our notation, let $\Psi = \sum_{j\in S_1}\cos\left(\frac{4\pi j}{m}\right)$ and $\Omega = \sum_{j\in S_1}\sin\left(\frac{4\pi j}{m}\right)$. Therefore, (2.7) becomes

$$\mathbf{F}_{S_1} = \frac{1}{m}\begin{bmatrix} n + \Psi & \Omega \\ \Omega & n - \Psi \end{bmatrix}.$$

Then

$$
\mathbf{F}_{S_1} - \lambda I = \begin{bmatrix} \frac{n}{m} + \frac{1}{m}\Psi - \lambda & \frac{1}{m}\Omega \\ \\ \frac{1}{m}\Omega & \frac{n}{m} - \frac{1}{m}\Psi - \lambda \end{bmatrix}
$$

$$
= \frac{1}{m} \begin{bmatrix} n + \Psi - m\lambda & \Omega \\ \\ \Omega & n - \Psi - m\lambda \end{bmatrix}, \tag{2.10}
$$

Which has characteristic equation

$$
\begin{vmatrix} n + \Psi - m\lambda & \Omega \\ \\ \Omega & n - \Psi - m\lambda \end{vmatrix} = 0.
$$

This yields the corresponding quadratic equation in λ:

$$
\begin{aligned}
(n - m\lambda)^2 - \Psi^2 - \Omega^2 &= n^2 - 2mn\lambda + m^2\lambda^2 - (\Psi^2 + \Omega^2) \\
&= m^2\lambda^2 - 2mn\lambda + n^2 - (\Psi^2 + \Omega^2) = 0.
\end{aligned}
$$

Denote λ_1, λ_2 to be the solutions of this equation. It follows that

$$
\|\mathbf{F}_{S_1}\| = \max\{\lambda_1, \lambda_2\} = \frac{n}{m} + \frac{\sqrt{\Psi^2 + \Omega^2}}{m}. \tag{2.11}
$$

By the same analysis, we obtain

$$
\|\mathbf{F}_{S_2}\| = \frac{p}{m} + \frac{\sqrt{\Psi^2 + \Omega^2}}{m}. \tag{2.12}
$$

It appears that the solution to our norm depends on the discriminant in (2.11) and (2.12):

$$\Psi^2 + \Omega^2 = \left(\sum_{j \in S_1} \cos\left(\frac{4\pi j}{m}\right)\right)^2 + \left(\sum_{j \in S_1} \sin\left(\frac{4\pi j}{m}\right)\right)^2. \tag{2.13}$$

In the next several sections we will examine how to simplify (2.13) for special cases of partitions.

Two-Partitions: $|S1| = |S2|$. Suppose our frame has the following restrictions:

- The frame $\mathcal{F}$ contains an *even* number of vectors, $|\mathcal{F}| = 2q$ where $q \in \mathbb{Z}^+$.
- We partition $[m]$ into two equal subsets such that $[m] = S_1 \cup S_2$, where $S_1 = \{1, 2, 3, ..., \frac{m}{2}\}$ and $S_2 = \{\frac{m}{2} + 1, \frac{m}{2} + 2, ..., m\} \Rightarrow |S_1| = |S_2| = n = \frac{m}{2}$.
- The elements in S_1 and S_2 correspond to consecutive vectors in our frame.

Essentially, S_1 corresponds to the first half of $\mathcal{F}$ and S_2 corresponds to the second half of $\mathcal{F}$. We are going to calculate the norm of the tensor product sums corresponding to the subsets S_1 and S_2. With these restrictions described above (2.13) becomes

$$\Psi^2 + \Omega^2 = \left(\sum_{j=1}^{m/2} \cos\left(\frac{4\pi j}{m}\right)\right)^2 + \left(\sum_{j=1}^{m/2} \sin\left(\frac{4\pi j}{m}\right)\right)^2. \tag{2.14}$$

What are these summations? They do not appear trivial. Let us closely examine (B.1) by replacing $m-1$ and y with $\frac{m}{2}$ and $\frac{4\pi}{m}$, respectively. Then

$$\begin{aligned}
\sum_{j=1}^{m/2} \sin\left(\frac{4\pi j}{m}\right) &= \sum_{j=0}^{m/2} \sin\left(\frac{4\pi j}{m}\right) \\
&= \sin\left(\frac{m}{4}\cdot\frac{4\pi}{m}\right)\sin\left(\frac{(m+2)}{4}\cdot\frac{4\pi}{m}\right)\frac{1}{\sin\left(\frac{2\pi}{m}\right)} \\
&= \sin\pi\sin\left(\frac{(m+2)\pi}{m}\right)\frac{1}{\sin\left(\frac{2\pi}{m}\right)} \\
&= 0. \qquad (2.15)
\end{aligned}$$

We can repeat the analysis on the cosine term utilizing (B.2) to get

$$\begin{aligned}
\sum_{j=0}^{m/2} \cos\left(\frac{4\pi j}{m}\right) &= 1 + \sum_{j=1}^{m/2} \cos\left(\frac{4\pi j}{m}\right) \\
&= \cos\left(\frac{m}{4}\cdot\frac{4\pi}{m}\right)\sin\left(\frac{(m+2)}{4}\cdot\frac{4\pi}{m}\right)\frac{1}{\sin\left(\frac{2\pi}{m}\right)} \\
&= -\sin\left(\pi+\frac{2\pi}{m}\right)\frac{1}{\sin\left(\frac{2\pi}{m}\right)} \\
&= -\left(\sin\pi\cos\left(\frac{2\pi}{m}\right)+\cos\pi\sin\left(\frac{2\pi}{m}\right)\right)\frac{1}{\sin\left(\frac{2\pi}{m}\right)} = 1
\end{aligned}$$

$$\Rightarrow \sum_{j=1}^{m/2} \cos\left(\frac{4\pi j}{m}\right) = 0. \qquad (2.16)$$

We substitute (2.15) and (2.16) back into (2.14) to get that $\Psi^2 + \Omega^2 = 0$. Therefore the norm described in (2.11) becomes

$$\|\mathbf{F}_{S_1}\| = \frac{n}{m}.$$

Recall one of the restrictions is that $n = \frac{m}{2}$; therefore, if we substitute this into the line above then we validate (2.9) which is

$$\|\mathbf{F}_{S_1}\| = \frac{1}{2} \leq \left(\frac{1}{\sqrt{2}} + \sqrt{\frac{2}{m}} \right)^2 .$$

Therefore, a subset formed by the first $m/2$ frame vectors is valid for (1.2). But what about the second subset S_2 which represents the second half of the frame? Since we are considering $S_2 = \{\frac{m}{2}+1, \frac{m}{2}+2, ..., m\}$, our discriminant (2.13) becomes

$$\Psi^2 + \Omega^2 = \left(\sum_{j=m/2+1}^{m} \cos\left(\frac{4\pi j}{m}\right) \right)^2 + \left(\sum_{j=m/2+1}^{m} \sin\left(\frac{4\pi j}{m}\right) \right)^2 . \tag{2.17}$$

We arrive at a similar question, what are these summations? If we examine the cosine summation in (2.17) we have

$$\sum_{j=m/2+1}^{m} \cos\left(\frac{4\pi j}{m}\right) = \sum_{j=1}^{m} \cos\left(\frac{4\pi j}{m}\right) - \sum_{j=1}^{m/2} \cos\left(\frac{4\pi j}{m}\right) . \tag{2.18}$$

Notice that the two summations on the right hand side of (2.18) were already calculated in (2.16) and (B.4); hence,

$$\sum_{j=m/2+1}^{m} \cos\left(\frac{4\pi j}{m}\right) = 0. \tag{2.19}$$

We can follow the same analysis for the sine summation in (2.17) and arrive at,

$$\sum_{j=m/2+1}^{m} \sin\left(\frac{4\pi j}{m}\right) = 0. \tag{2.20}$$

Then we substitute (2.19) and (2.20) back into (2.17) to get the same result for the discriminant $\Psi^2 + \Omega^2 = 0$. Therefore, the norm corresponding to the second subset S_2 will also be $\frac{1}{2}$, and we have

$$\|\mathbf{F}_{S_1}\| = \|\mathbf{F}_{S_2}\| = \frac{1}{2} \leq \left(\frac{1}{\sqrt{2}} + \sqrt{\frac{2}{m}}\right)^2.$$

From here we can conclude that given an *even* number of vectors that form an equinorm-equiangular frame $\mathcal{F}$ then we can always partition the frame into two subsets with equal cardinality. The subsets are such that they correspond to consecutive vectors in our frame $\mathcal{F}$ and the set $\{S_1, S_2\}$ is a valid partition for (1.2).

Two-Partitions: $|S_1| \neq |S_2|$. Let us examine another partition with the following qualities:

- The cardinalities of S_1 and S_2 are *not* equal; therefore, $|S_1| = n$ and $|S_2| = p$ with $n + p = m$ and $n \neq p$.

- We will consider the "nice" case when the indices in S_1 and S_2 are consecutive, $S_1 = \{1, 2, 3, ..., n\}$ and $S_2 = \{n+1, n+2, n+3, ..., m-1, m\}$.

- There are any number of vectors in our frame $\mathcal{F}$, where $|\mathcal{F}| = m$.

As in the previous section, the problem reduces to determining the value of the discriminant (2.13). The following two subsections are dedicated to finding closed form equations describing the norms of the matrices corresponding to the outer products of frame vectors. The first subsection analyzes S_1 and the second analyzes S_2.

The Subset S_1. Our goal is to compute the discriminant corresponding to the subset S_1. Thus we have

$$\Psi^2 + \Omega^2 = \left(\sum_{j=1}^{n} \cos\left(\frac{4\pi j}{m}\right)\right)^2 + \left(\sum_{j=1}^{n} \sin\left(\frac{4\pi j}{m}\right)\right)^2. \tag{2.21}$$

We begin by using the series (B.1) and (B.2) to get

$$\sum_{j=1}^{n} \sin\left(\frac{4\pi j}{m}\right) = \sin\left(\frac{2\pi(n-1)}{m}\right) \sin\left(\frac{2\pi n}{m}\right) \frac{1}{\sin\left(\frac{2\pi}{m}\right)} \tag{2.22}$$

and

$$\sum_{j=1}^{n} \cos\left(\frac{4\pi j}{m}\right) = \cos\left(\frac{2\pi(n-1)}{m}\right) \sin\left(\frac{2\pi n}{m}\right) \frac{1}{\sin\left(\frac{2\pi}{m}\right)}. \tag{2.23}$$

We substitute (2.22) and (2.23) back into (2.21) to get

$$\Psi^2 + \Omega^2 = \sin^2\left(\frac{2\pi n}{m}\right)\frac{1}{\sin^2\left(\frac{2\pi}{m}\right)}\left(\cos^2\left(\frac{2\pi(n-1)}{m}\right) + \sin^2\left(\frac{2\pi(n-1)}{m}\right)\right)$$

$$= \sin^2\left(\frac{2\pi n}{m}\right)\frac{1}{\sin^2\left(\frac{2\pi}{m}\right)}. \tag{2.24}$$

We substitute this result into the norm described by (2.11) and we have

$$\|\mathbf{F}_{S_1}\| = \frac{n}{m} + \frac{\sin\left(\frac{2\pi n}{m}\right)}{m\sin\left(\frac{2\pi}{m}\right)}. \tag{2.25}$$

Interestingly, this norm depends only on the number of vectors n in the subset S_1.

The Subset S_2. We still need to examine the second subset $S_2 = \{n+1, n+2, ...m-1, m\}$. Our discriminant from (2.13) becomes

$$\Psi^2 + \Omega^2 = \left(\sum_{j=n+1}^{m}\cos\left(\frac{4\pi j}{m}\right)\right)^2 + \left(\sum_{j=n+1}^{m}\sin\left(\frac{4\pi j}{m}\right)\right)^2. \tag{2.26}$$

This is not as *nice* as before, but we may utilize a few results involving trigonometric summations. If we utilize (2.22) and (B.3) we obtain

$$\sum_{j=1}^{m} \sin\left(\frac{4\pi j}{m}\right) = \sum_{j=1}^{n} \sin\left(\frac{4\pi j}{m}\right) + \sum_{j=n+1}^{m} \sin\left(\frac{4\pi j}{m}\right)$$

$$0 = \sin\left(\frac{2\pi(n-1)}{m}\right)\sin\left(\frac{2\pi n}{m}\right)\frac{1}{\sin\left(\frac{2\pi}{m}\right)} + \sum_{j=n+1}^{m} \sin\left(\frac{4\pi j}{m}\right)$$

$$\Rightarrow \sum_{j=n+1}^{m} \sin\left(\frac{4\pi j}{m}\right) = -\sin\left(\frac{2\pi(n-1)}{m}\right)\sin\left(\frac{2\pi n}{m}\right)\frac{1}{\sin\left(\frac{2\pi}{m}\right)}. \tag{2.27}$$

If we follow the same analysis on the cosine term of (2.26), then we get

$$\sum_{j=n+1}^{m} \cos\left(\frac{4\pi j}{m}\right) = -\cos\left(\frac{2\pi(n-1)}{m}\right)\sin\left(\frac{2\pi n}{m}\right)\frac{1}{\sin\left(\frac{2\pi}{m}\right)}. \tag{2.28}$$

Substitute (2.28) and (2.27) into (2.26) and we arrive at the same result for the subset S_2, the discriminant (2.24) becomes

$$\Psi^2 + \Omega^2 = \sin^2\left(\frac{2\pi n}{m}\right)\frac{1}{\sin^2\left(\frac{2\pi}{m}\right)}.$$

With $p = m - n$, (2.12) becomes

$$\|\mathbf{F}_{S_2}\| = \frac{m-n}{m} + \frac{\sin\left(\frac{2\pi n}{m}\right)}{m\sin\left(\frac{2\pi}{m}\right)}. \tag{2.29}$$

This norm may also be described in terms $\|\mathbf{F}_{S_1}\|$:

$$\begin{aligned} \|\mathbf{F}_{S_2}\| &= 1 - \frac{2n}{m} + \|\mathbf{F}_{S_1}\| \\ &= \frac{p-n}{m} + \|\mathbf{F}_{S_1}\|. \end{aligned} \tag{2.30}$$

If we let $n \geq p$, then $\|\mathbf{F}_{S_1}\| \geq \|\mathbf{F}_{S_2}\|$. Notice that equality occurs if we have an even number of vectors in our frame, and we would arrive at the same conclusion from the section on the $|S_1| = |S_2|$ partition.

We have thus found closed form equations for the norms of the partitioned frame vector sets S_1 and S_2. However, we need to explore how do the norms corresponding to these subsets behave in relation to the conclusion of the corollary (1.2)? In the subsequent sections we will analyze the behavior of our matrix norm equations corresponding to the subsets S_1 and S_2. From there we shall identify some ways in determining valid partitions of equinorm-equiangular frames for the case where $|S_1| \neq |S_2|$.

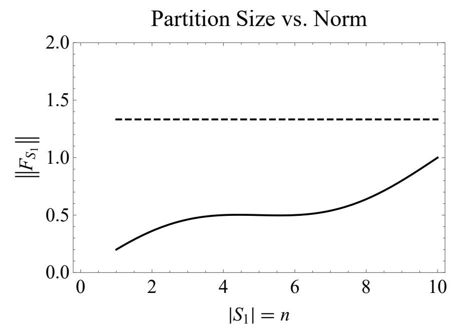
Matrix Norm Analysis. In this section we will analyze the behavior of the norm equations derived in the previous two sections. Since we are only forming two subsets of the frame and they are not equal in size, this results in one subset containing more than half the frame elements and the other contains less than half. Hence, with out loss of generality we will let $|S_1| = n > \left\lfloor \frac{|\mathcal{F}|}{2} \right\rfloor$ and as a consequence $|S_2| = p < n$. Our focus will be on analyzing the norms corresponding to the subset S_1 because if $n > p$ then $\|\mathbf{F}_{S_1}\| > \|\mathbf{F}_{S_2}\|$. If we can find a way to obtain the subset S_1, then S_2 is simply its complement.

In Table 2.1 there are four figures, each figure is showing how the norm (corresponding to the subset S_1) compares to the corollary bound for various cardinalities of S_1. The horizontal axis represents the number of vectors in the subset, $|S_1| = n$, and the vertical axis is the value of the norm. The horizontal dashed line (which is constant since it only depends on m) is the corollary bound from (1.2) and the curve is the function described by (2.25).

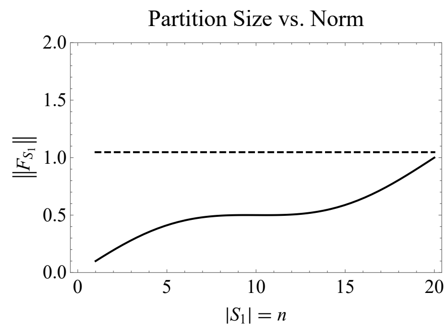
Notice in figures 2.1a and 2.1b no matter the size of the subset S_1, the norm always remains less than the bound described by the corollary (1.2). On the other hand, in figures 2.1c and 2.1d the norms fail when the size of S_1 is sufficiently large. Thus it suffices to find the largest value for n where the norm satisfies the corollary bound? I will refer to this special value as n_{max} and it will be the size of the largest S_1 subset of an m vector frame just before we fail the corollary bound. Visually, in the figures n_{max} is the intersection of the horizontal line and the curve, assuming they intersect at all.

Table 1

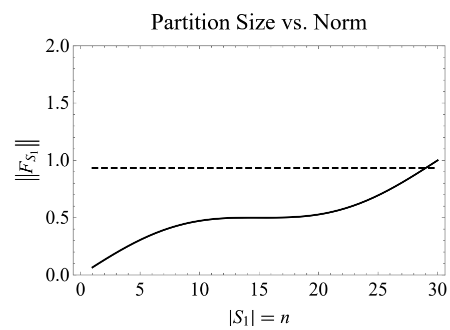
Comparison of frame matrix norms to Corollary 1.5



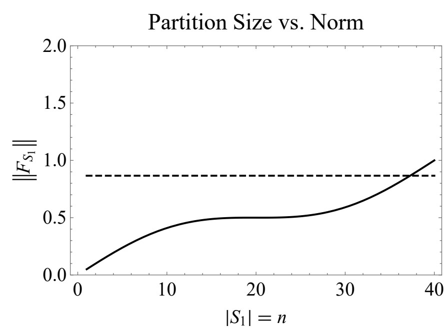
(A) $m = 10$ frame vectors



(B) $m = 20$ frame vectors



(C) $m = 30$ frame vectors



(D) $m = 40$ frame vectors

Below is another figure demonstrating an extreme case. Notice that for such a large m value it appears n_{max} is roughly half of m.

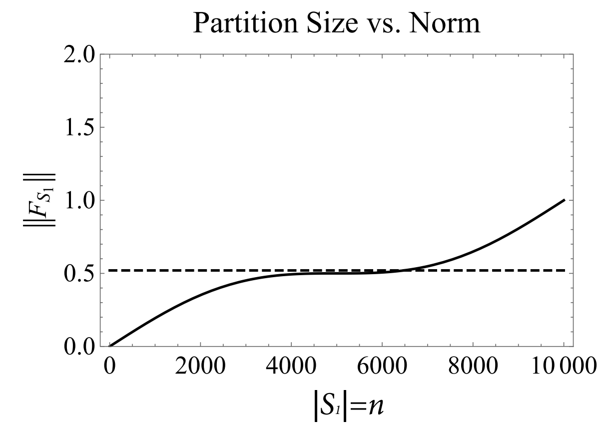


Figure 2. *m* = 10000 frame vectors.

Table 2 is a table of values for various sizes of finite frames, m, and n_{max}. For instance, if we examine the third row when $m = 40$ we have a subset containing 37 vectors and it is valid with the corollary, but as soon as we go beyond to 38 vectors in the subset then we fail the corollary. Hence, $n_{max} = 37$ for a frame consisting of 40 vectors.

Table 2

Max Subset Size (n_{max}) Until Failure

$\|\mathcal{F}\| = m$	n_{max}	$\|\mathbf{F}_{S_1}\|$	Cor.-1.5	n_{max}/m
30	28	0.868123...	0.931815...	0.933333
40	37	0.852447...	0.866228...	0.925
50	45	0.806204...	0.822843...	0.9
60	53	0.776643...	0.791532...	0.883333...
70	61	0.756238...	0.767617...	0.871429...
100	84	0.705533...	0.72	0.84
200	160	0.64861...	0.651421...	0.8
300	233	0.619729...	0.622137...	0.776667...
400	305	0.603829...	0.605	0.7625
500	376	0.592853...	0.593443...	0.75
600	446	0.584315...	0.584983...	0.743333...
1000	721	0.564478...	0.565245...	0.721
10000	6474	0.520186...	0.5202...	0.6474
100000	59943	0.5063428...	0.50634455...	0.59943
1000000	567460	0.50200191...	0.502002...	0.56746
10000000	5458762	0.500632654...	0.500632655...	0.54 58762

Interestingly, when there are 23 or fewer vectors contained in the frame any size of S_1 will be valid. Once we go beyond 23 vectors in a frame then we need to solve an equation in order to find n_{max}. Recall that for extremely large m it appears that the largest possible subset size approaches $n_{max} = m/2$. This is easily shown by taking the limit of (1.2):

$$\lim_{m\to\infty}\left(\frac{1}{\sqrt{2}}+\sqrt{\frac{2}{m}}\right)^2 = \frac{1}{2}.$$

Next, we solve the equation

$$\|\mathbf{F}_{S_1}\| = \left(\frac{1}{\sqrt{2}}+\sqrt{\frac{2}{m}}\right)^2.$$

This equation can be written in terms of the equation for $\|\mathbf{F}_{S_1}\|$ (2.25) to get

$$\frac{x}{m}+\frac{\sin\left(\frac{2\pi x}{m}\right)}{m\sin\left(\frac{2\pi}{m}\right)} = \left(\frac{1}{\sqrt{2}}+\sqrt{\frac{2}{m}}\right)^2. \tag{2.31}$$

The solution x is related to n_{max} in the following way

$$n_{max} = \lfloor x \rfloor.$$

Equation (2.31) is not elementary, and we use Mathematica to solve this equation for x, and therefore n_{max}, for various values of m in Table 2. In conclusion, a *valid* partition $\{S_1, S_2\}$ of $[m]$, where $|S_1| \neq |S_2|$, can always be formed if $|S_1| \leq n_{max}$.

Almost even split partition. Let us examine another partition, this time we have the following criteria:

- $|\mathcal{F}|$ is odd or $|\mathcal{F}| = 2q + 1$ where $q \in \mathbb{Z}^+$.
- Let $|S_1| = n = \frac{m+1}{2}$ and $|S_2| = p = \frac{m-1}{2}$.

We make the above substitutions into (2.25) to get

$$\begin{aligned} \|\mathbf{F}_{S_1}\| &= \frac{m+1}{2m} + \frac{\sin\left(\frac{\pi(m+1)}{m}\right)}{m \sin\left(\frac{2\pi}{m}\right)} \\ &= \frac{1}{2} + \frac{1}{2m} + \frac{\sin\left(\pi + \frac{\pi}{m}\right)}{2m \sin\left(\frac{\pi}{m}\right) \cos\left(\frac{\pi}{m}\right)} \\ &= \frac{1}{2} + \frac{1}{2m} - \frac{1}{2m \cos\left(\frac{\pi}{m}\right)}. \end{aligned}$$

Next we need to determine when this quantity less than the bound (1.2). Let us examine the following inequality problem:

$$\begin{aligned} &\frac{1}{2} + \frac{1}{2m} - \frac{1}{2m \cos\left(\frac{\pi}{m}\right)} \leq \frac{1}{2} + \frac{2}{\sqrt{m}} + \frac{2}{m} \\ &\Rightarrow \frac{1}{2m} - \frac{1}{2m \cos\left(\frac{\pi}{m}\right)} \leq \frac{2}{\sqrt{m}} + \frac{2}{m} \\ &\Rightarrow \frac{1}{2} - \frac{1}{2 \cos\left(\frac{\pi}{m}\right)} \leq 2\sqrt{m} + 2 \\ &\Rightarrow -\frac{1}{2 \cos\left(\frac{\pi}{m}\right)} \leq 2\sqrt{m} + \frac{3}{2}. \end{aligned} \tag{2.32}$$

The inequality (2.32) holds for all positive integers m.We can follow this same procedure if we substitute $n = \frac{m-1}{2}$ into (2.29). Then (2.32) becomes

$$-\frac{1}{2\cos\left(\frac{\pi}{m}\right)} \leq 2\sqrt{m} + \frac{5}{2}$$

and we arrive at the same conclusion as before. Therefore, given a frame $\mathcal{F}$ with an odd number of vectors we can always a form a valid partition $\{S_1, S_2\}$ of $[m]$ such that $|S_1| = \frac{m+1}{2}$ and $|S_2| = \frac{m-1}{2}$.

Chapter 3

Grassmannian Frames

Grassmannian Frames

Grassmanian frames are a class of frames that satisfy a condition involving *maximal frame correlation*. The defintion provided by Strohmer and Heath [14] of maximal frame correlation is the following:

Definition 3.1. *For a given unit norm frame* $\{f_j\}_{j=1}^n$ *in* $\mathbb{E}^m$ *we define the maximal frame correlation* $\mathcal{M}\left(\{f_j\}_{j=1}^n\right)$ *by*

$$\mathcal{M}\left(\{f_j\}_{j=1}^n\right) = \max_{j,k,j\neq k}\{|\langle f_j, f_k\rangle|\}. \tag{3.1}$$

As an example suppose we have a unit frame with three vectors $\{v_1, v_2, v_3\} \in \mathbb{R}^2$. We can think of the *maximal correlation* of this set of vectors as the largest magnitude of their dot products. Suppose $\langle \star, \star \rangle$ denotes the dot product of two vectors and that $\langle v_1, v_2\rangle = -0.8$, $\langle v_1, v_3\rangle = 1$, and $\langle v_2, v_3\rangle = 0.5$. Then the maximal frame correlation $\mathcal{M}\left(\{v_1, v_2, v_3\}\right) = 1$. As a side note if the set of vectors are orthonormal then the maximal correlation is 0. The Grassmannian frame is defined as follows [14]:

Definition 3.2. *A sequence of vectors* $\{u_j\}_{j=1}^n$ *in* $\mathbb{E}^m$ *is called a Grassmannian frame if it is the solution to*

$$\min\{\mathcal{M}\left(\{f_j\}_{j=1}^n\right)\} \tag{3.2}$$

where the minimum is taken over all unit norm frames $\{f_j\}_{j=1}^n$ *in* $\mathbb{E}^m$.

In other words, if we have a set of finite frames, we find the *smallest* of the maximal frame correlations. Then a frame that has a maximal frame equal to this is a Grassmannian frame. For our purposes we will be working in $\mathbb{C}^m$.

In the subsequent sections we will go through the construction of Grassmannian frames using skew-symmetric conference matrices. Then we will demonstrate that the *optimal* Grassmannian frames meet the hypothesis of Corollary 1.5. We will also examine the inner products of Grassmannian frame vectors. The last section will be dedicated to examining the partitions for Grassmannian frames for various frame sizes.

Constructing Optimal Grassmannian Frames

We will explore *optimal* Grassmannian frames and this occurs when $n = 2m$. There-fore, the minimum of the maximal correlations is

$$\min\{\mathcal{M}\left(\{f_j\}_{j=1}^n\right)\} = \sqrt{\frac{n-m}{m(n-1)}} = \frac{1}{\sqrt{2m-1}}.$$

Strohmer and Heath give a construction of *optimal* Grassmannian frames [14] using a recursive algorithm. We first begin with a 2×2 skew-symmetric conference matrix

$$\mathbf{C}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and define higher order ones by the recurrence

$$\mathbf{C}_{2m} = \begin{bmatrix} \mathbf{C}_m & \mathbf{C}_m - I_m \\ \mathbf{C}_m + I_m & -\mathbf{C}_m \end{bmatrix}.$$

Then we define

$$\mathbf{R} := \mathbf{R}_{2m} = \alpha \mathbf{C}_{2m} + I, \tag{3.3}$$

where $\alpha = \frac{i}{\sqrt{2m-1}}$ and $n = 2m$. Notice that this algorithm will generate conference matrices that are sizes of 2, 4, 8, 16, and so on, which are powers of 2. As an example if we let $m = 2$, then $n = 4$ and we will have the 4×4 conference matrix

$$\mathbf{C}_4 = \mathbf{C}_{2\cdot 2} = \begin{bmatrix} \mathbf{C}_2 & \mathbf{C}_2 - I_2 \\ \mathbf{C}_2 + I_2 & -\mathbf{C}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}.$$

Therefore, $\alpha = \frac{i}{\sqrt{3}}$ and our $\mathbf{R}$-matrix is

$$\mathbf{R} = \begin{bmatrix} 1 & -\frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & 1 & \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & 1 & \frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & 1 \end{bmatrix}.$$

Next we take our $\mathbf{R}$-matrix and compute the spectral factorization

$$\mathbf{R} = \mathbf{W}\mathbf{\Lambda}\mathbf{W}^*, \tag{3.4}$$

where $\mathbf{W}$ is a unitary matrix with columns that are the eigen-vectors of $\mathbf{R}$ and $\boldsymbol{\Lambda}$ is a $n \times n$ diagonal matrix with diagonal entries being the eigenvalues of the $\mathbf{R}$-matrix. For this thesis we are going to analyze optimal Grassmannian frames and let $n = 2m$. From Strohmer and Heath's construction it is known that the eigenvalues are 0 and 2, each with multiplicity m. Thus $\boldsymbol{\Lambda}$ is a diagonal matrix that has m 2's in the left half and m 0's in the right half. Going back to our example when $n = 4$ we have

$$\boldsymbol{\Lambda} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Next we define the frame vectors f_k to be

$$\begin{aligned} f_k &= \sqrt{\frac{n}{m}}\{\mathbf{W}_{k,l}\}_{l=1}^{m} \\ &= \sqrt{2}\{\mathbf{W}_{k,l}\}_{l=1}^{m}. \end{aligned} \tag{3.5}$$

Our frame $\mathcal{F}$ is given by

$$\mathcal{F} = \{f_k\}_{k=1}^{n}. \tag{3.6}$$

In essence, we take the $\mathbf{W}$-matrix (with eigen-vectors as its columns) and isolate the first m columns corresponding to the eigenvalue $\lambda = 2$. We then take the rows of these first m columns and multiply them by the coefficient $\sqrt{2}$. These rows constitute the frame vectors f_k. Notice that,

$$\begin{aligned}
\mathbf{R} &= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \omega_1 & \omega_2 & \cdots & \omega_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} 2 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} \leftarrow & \overline{\omega}_1^T & \rightarrow \\ \leftarrow & \overline{\omega}_2^T & \rightarrow \\ & \vdots & \\ \leftarrow & \overline{\omega}_n^T & \rightarrow \end{bmatrix} \\
&= 2\omega_1\omega_1^* + 2\omega_2\omega_2^* + \cdots + 2\omega_m\omega_m^* + \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} \\
&= 2\sum_{k=1}^{m} \omega_k\omega_k^*. \qquad (3.7)
\end{aligned}$$

This verifies that our frame need only constitute vectors from the left half of $\mathbf{W}$. Notice that a bit of dimension analysis verifies (3.7), namely

$$[n \times n] = [n \times n][n \times n][n \times n].$$

Grassmanian Frames and the Hypothesis of Corollary 1.5

In this section we will prove that Grassmanian frames meet the hypothesis of Corollary 1.5 if we omit the coefficient $\sqrt{2}$ from our original definition (3 .5). Let $\omega_k \in \mathbb{C}^n$ be the eigen-vectors of $\mathbf{R}$ and $\phi_j \in \mathbb{C}^m$ be the rows of $\mathbb{W}$, where $\mathbb{W}$ consists of the vectors ω_k as columns:

$$\mathbb{W} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \omega_1 & \omega_2 & \cdots & \omega_m \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \leftarrow & \phi_1^T & \rightarrow \\ \leftarrow & \phi_2^T & \rightarrow \\ & \vdots & \\ \leftarrow & \phi_n^T & \rightarrow \end{bmatrix}$$

and (3.7) becomes

$$\mathbf{R} = 2\mathbb{W}\mathbb{W}^*. \tag{3.8}$$

Again with some dimension analysis we have

$$[n \times n] = [n \times m][m \times n].$$

Recall that $\mathbf{W}$ was computed from the spectral factorization of $\mathbf{R}$, and $\mathbf{W}$ is unitary. This means $\mathbf{W}\mathbf{W}^* = \mathbf{W}^*\mathbf{W} = I$ and the columns and rows of $\mathbf{W}$ form an orthonormal basis. Since $\mathbb{W}$ consists of the first m columns of $\mathbf{W}$, then the columns of $\mathbb{W}$ are orthonormal. If we utilize the *standard orthonormal* vectors in n-space we can re-express our frame vectors from (3.5) as

$$f_k = \sqrt{2}\mathbb{W}_k = \sqrt{2}\mathbb{W}^T e_k = \sqrt{2}\phi_k.$$

Hence

$$\begin{aligned}
f_1 &= \sqrt{2}\mathbb{W}^T e_1 = \sqrt{2}\phi_1 \\
f_2 &= \sqrt{2}\mathbb{W}^T e_2 = \sqrt{2}\phi_2 \\
&\vdots \\
f_n &= \sqrt{2}\mathbb{W}^T e_n = \sqrt{2}\phi_n.
\end{aligned}$$

If we now attempt to verify the hypothesis of the corollary (1.1), then we get

$$
\begin{aligned}
\sum_{k=1}^{n} f_k f_k^* &= 2\sum_{k=1}^{n} \mathbb{W}^T e_k \left(\mathbb{W}^T e_k\right)^* \\
&= 2\sum_{k=1}^{n} \mathbb{W}^T e_k e_k^T \overline{\mathbb{W}} \\
&= 2\mathbb{W}^T \left(\sum_{k=1}^{n} e_k e_k^T\right) \overline{\mathbb{W}} \\
&= 2\mathbb{W}^T I^{n\times n} \overline{\mathbb{W}} \\
&= 2\begin{bmatrix} \leftarrow & \omega_1^T & \rightarrow \\ \leftarrow & \omega_2^T & \rightarrow \\ & \vdots & \\ \leftarrow & \omega_m^T & \rightarrow \end{bmatrix}
\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}
\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \overline{\omega}_1 & \overline{\omega}_2 & \cdots & \overline{\omega}_m \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \\
&= 2\begin{bmatrix} \omega_1^T \overline{\omega}_1 & 0 & \cdots & 0 & 0 \\ 0 & \omega_2^T \overline{\omega}_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \omega_{m-1}^T \overline{\omega}_{m-1} & 0 \\ 0 & 0 & \cdots & 0 & \omega_m^T \overline{\omega}_m \end{bmatrix} \\
&= 2I^{m\times m}.
\end{aligned}
$$

Therefore, our frame does not quite satisfy the hypothesis (1.1), but this can be easily rectified by rescaling each f_k by a factor of $\frac{1}{\sqrt{2}}$:

$$f_k = \{\mathbf{W}_{k,l}\}_{l=1}^{m} = \mathbb{W}_k = \mathbb{W}^T e_k = \phi_k \Rightarrow \mathcal{F} = \{f_k\}_{k=1}^{n}. \tag{3.9}$$

If we now repeat the same analysis as above we get

$$\sum_{k=1}^{n} f_k f_k = I^{m\times m}.$$

Hence our redefined frame (3.9) satisfies the hypothesis (1.1) in Corollary 1.5.

Inner Products of Grassmannian Frame Vectors

In this section we will analyze the inner products of the vectors that are formed in $\mathbf{W}$ and $\mathbb{W}$. Recall that since we have the spectral decomposition of $\mathbf{R} = \mathbf{W\Lambda W}^* = 2\mathbb{W}\mathbb{W}^*$, then the columns of $\mathbf{W}$ and $\mathbb{W}$ constitute an orthonormal basis, i.e., for all $k, j \in [n]$,

$$\langle \omega_j, \omega_k \rangle = \begin{cases} 1 & j = k; \\ 0 & j \neq k. \end{cases} \tag{3.10}$$

So clearly the columns of $\mathbb{W}$ are normal and orthogonal, but we are interested in the rows of $\mathbb{W}$ because this constitutes our frame $\mathcal{F}$ which we defined in (3.9). Notice that we can express each row of $\mathbb{W}$ using our standard orthonormal vectors

$$\phi_k = \mathbb{W}^T e_k. \tag{3.11}$$

We will now derive the norm of the vector ϕ_k:

$$\begin{aligned}
\|\phi_k\|^2 &= \phi_k^* \phi_k \\
&= \left(\mathbb{W}^T e_k\right)^* \mathbb{W}^T e_k \\
&= e_k^T \overline{\mathbb{W}} \mathbb{W}^T e_k \\
&= \frac{1}{2} e_k^T \mathbf{R}^T e_k = \frac{1}{2}.
\end{aligned}$$

Therefore

$$\|\phi_k\| = \frac{1}{\sqrt{2}}. \tag{3.12}$$

More generally the inner product of ϕ_k, ϕ_j is given by

$$\begin{aligned}
\phi_k^* \phi_j &= \left(\mathbb{W}^T e_k\right)^* \mathbb{W}^T e_j \\
&= e_k^T \overline{\mathbb{W}} \mathbb{W}^T e_j \\
&= \frac{1}{2} e_k^T \mathbf{R}^T e_j.
\end{aligned}$$

This extracts an entry that is off the diagonal of $\mathbf{R}$. Hence,

$$\phi_k^* \phi_j = \frac{\pm i}{2\sqrt{n-1}}. \tag{3.13}$$

The combined results of (3.13) and (3.12) and (3.9) yield

$$\langle f_j, f_k \rangle = \langle \phi_j, \phi_k \rangle = \begin{cases} \frac{1}{2} & j = k; \\ \frac{\pm i}{2\sqrt{n-1}} & j \neq k. \end{cases} \tag{3.14}$$

Recall that the vectors ϕ_k are the rows of $\mathbb{W}$, which are our frame vectors f_k. More importantly notice that our frame vectors are not orthogonal. This means our calculation for the norm of the outer product sum will be more involved.

Norms of Outer Products of Grassmannian Frame Vectors (2 Vectors)

Now we will attempt to establish a closed formula for the norm of the outer prod-uct of the sum of frame vectors. First, if we utilize the result from (A.4) and (3.14) we have

$$\|f_k f_k^*\| = \|f_k\|^2 = \frac{1}{2}. \tag{3.15}$$

Hence the norm of the outer product of one frame vector has a nice closed form. Next we calculate the norm of the sum of two frame vectors:

$$\begin{aligned}\mathbf{F} = f_k f_k^* + f_j f_j^* &= \mathbb{W}^T e_k \left(\mathbb{W}^T e_k\right)^* + \mathbb{W}^T e_j \left(\mathbb{W}^T e_j\right)^* \\ &= \mathbb{W}^T e_k e_k^T \overline{\mathbb{W}} + \mathbb{W}^T e_j e_j^T \overline{\mathbb{W}} \\ &= \mathbb{W}^T \left(e_k e_k^T + e_j e_j^T\right) \overline{\mathbb{W}}. \end{aligned} \tag{3.16}$$

Observe that $\mathbf{F}$ is Hermitian, i.e., $\mathbf{F} = \mathbf{F}^*$. If we want to find $\|\mathbf{F}\|$, then we need to compute $\|\mathbf{F}x\|^2$ where x is any unit vector. Hence, we utilize the results of (A.8) to get

$$\|\mathbf{F}x\|^2 = x^* \left(\frac{1}{2} f_k f_k^* + f_k f_k^* f_j f_j^* + f_j f_j^* f_k f_k^* + \frac{1}{2} f_j f_j^*\right) x.$$

Remember that $f_k^* f_j$ is equal to the off diagonal entries of $\frac{1}{2}\mathbf{R}$ and $\mathbf{R}$ is Hermitian. Therefore

$$f_k^* f_j = -f_j^* f_k = \rho,$$

where $|\rho| = \frac{i}{2\sqrt{n-1}}$. We then have

$$\|\mathbf{F}x\|^2 = x^* \left(\frac{1}{2} f_k f_k^* + \rho \left(f_k f_j^* - f_j f_k^* \right) + \frac{1}{2} f_j f_j^* \right) x. \tag{3.17}$$

Observe that

$$f_k f_k^* = \mathbb{W}^T e_k e_k^T \overline{\mathbb{W}}$$

$$f_k f_j^* = \mathbb{W}^T e_k e_j^T \overline{\mathbb{W}}.$$

As a result, equation (3.17) becomes

$$\|\mathbf{F}x\|^2 = x^* \mathbb{W}^T \left(\frac{1}{2} e_k e_k^T + \rho \left(e_k e_j^T - e_j e_k^T \right) + \frac{1}{2} e_j e_j^T \right) \overline{\mathbb{W}} x. \tag{3.18}$$

While the equation above appears nice, it is not trivial to use Lagrange multipliers to solve for λ_{max}. We will examine some particular examples for $n = 4$, 8, 16, and 32 and utilize numerical analysis to determine what partitions are valid with respect to the Corollary 1.5.

Case *n* = 4. Let us examine the case when $n = 4$. This has the implication that there will be 4 frame vectors in $\mathbb{C}^2$. Below is the corresponding $\mathbf{R}$ matrix:

$$\mathbf{R} = \begin{bmatrix} 1 & -\frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & 1 & \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & 1 & \frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & 1 \end{bmatrix}. \tag{3.19}$$

An orthonormal basis for $\mathbf{W}$ can be chosen so that

$$\mathbf{W} = \begin{bmatrix} \frac{1}{6}\left(3i - \sqrt{3}\right) & \frac{i}{\sqrt{6}} & \frac{1}{6}\left(-3i - \sqrt{3}\right) & -\frac{i}{\sqrt{6}} \\ \frac{1}{6}\left(3i + \sqrt{3}\right) & -\frac{i}{\sqrt{6}} & \frac{1}{6}\left(-3i + \sqrt{3}\right) & \frac{i}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{6}} \end{bmatrix}, \tag{3.20}$$

where the columns of $\mathbf{W}$ are the eigenvectors of $\mathbf{R}$. We can then form our frame using (3.9) by simply extracting the first two columns of (3.20):

$$\mathbb{W} = \begin{bmatrix} \frac{1}{6}\left(3i - \sqrt{3}\right) & \frac{i}{\sqrt{6}} \\ \frac{1}{6}\left(3i + \sqrt{3}\right) & -\frac{i}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{6}} \end{bmatrix}. \tag{3.21}$$

Each row of $\mathbb{W}$ is a frame vector f_k; hence there are 4 vectors in $\mathbb{C}^2$:

$$\begin{aligned} \mathcal{F} &= \{f_k\}_{k=1}^4 \\ &= \left\{ \begin{bmatrix} \frac{1}{6}\left(3i - \sqrt{3}\right) \\ \frac{i}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{1}{6}\left(3i + \sqrt{3}\right) \\ -\frac{i}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{i}{\sqrt{6}} \end{bmatrix} \right\}. \end{aligned} \tag{3.22}$$

The norm of each frame vector is $\frac{1}{\sqrt{2}}$. Let us now describe our goal:

- The four frame vectors $\{f_1, f_2, f_3, f_4\}$ correspond to the four elements $[4] = \{1, 2, 3, 4\}$.

- We need to find a partition $\{S_1, ..., S_r\}$ of $[4]$ such that

$$S_1 \cup \cdots \cup S_r = [4],$$

  and the norms of the sum of outer products of frame vectors corresponding to each subset S_k is satisfies bound (1.2).

Let us examine some partitions. Recall the matrix $\mathbf{F}$ which is the sum of the outer products of vectors corresponding to a subset S_k.

Table 3

Subset size and the corresponding norms for n = 4 case.

$S_k = \{\cdots\}$	$\|\mathbf{F}\|$
$\{1\}$	0.5
$\{2\}$	0.5
$\{3\}$	0.5
$\{4\}$	0.5
$\{1, 2\}$	0.788675
$\{1, 3\}$	0.788675
$\{1, 4\}$	0.788675
$\{2, 3\}$	0.788675
$\{2, 4\}$	0.788675
$\{3, 4\}$	0.788675
$\{1, 2, 3\}$	1
$\{1, 2, 4\}$	1
$\{1, 3, 4\}$	1
$\{2, 3, 4\}$	1
$\{1, 2, 3, 4\}$	1

From Table 3 it appears that the norm of $\mathbf{F}$ takes on only certain values depending only on the size of S_k. It is clear from Corollary 1.5 that

$$1 \leq r \leq n = 4. \tag{3.23}$$

Recall that $\delta = \frac{1}{2}$. Thus we can manipulate (3.23) to get

$$\begin{aligned}
1 \leq r \leq 4 &\Rightarrow 1 \leq \sqrt{r} \leq 2 \\
&\Rightarrow 1 \geq \frac{1}{\sqrt{r}} \geq \frac{1}{2} \\
&\Rightarrow 1 + \frac{1}{\sqrt{2}} \geq \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{2}} \geq \frac{1}{2} + \frac{1}{\sqrt{2}} \\
&\Rightarrow \left(1 + \frac{1}{\sqrt{2}}\right)^2 \geq \left(\frac{1}{\sqrt{r}} + \frac{1}{\sqrt{2}}\right)^2 \geq \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)^2 \\
&\Rightarrow \frac{3}{2} + \sqrt{2} \geq \left(\frac{1}{\sqrt{r}} + \frac{1}{\sqrt{2}}\right)^2 \geq \frac{3}{4} + \frac{\sqrt{2}}{2} = 1.457106... \qquad (3.24)
\end{aligned}$$

This shows that the corollary bound is greater than unity for all r. If we compare this to Table 3.1 then any subset corresponding to our frame $\mathcal{F}$ will satisfy (1.2) for the case $n = 4$.

An analytic approach for $n = 4$. We compute analytically the norm of $\mathbf{F}$ for a subset $S_1 = \{1, 2\}$ corresponding to the two vectors f_1, f_2. Recall that each frame vector f_k is obtained from a row of $\mathbb{W}$. Therefore

$$f_k = \begin{bmatrix} w_{k,1} \\ w_{k,2} \end{bmatrix}$$

Suppose we rotate f_1 by a unitary matrix $\mathbf{U}$ to obtain $\tilde{f}_1 = \mathbf{U}f_1 = \{a, 0\}^T$ so that

$$\|\tilde{f}_1\|^2 = a\overline{a} = \|f_1\|^2 = \frac{1}{2}.$$

Then $\tilde{f}_2 = \mathbf{U}f_2 = \{b, c\}^T$ and

$$\|\tilde{f}_2\|^2 = b\overline{b} + c\overline{c} = \|f_1\|^2 = \frac{1}{2}.$$

But notice that $\langle \tilde{f}_1, \tilde{f}_2 \rangle = \{a, 0\} \cdot \{\overline{b}, \overline{c}\} = a\overline{b} = \langle f_1, f_2 \rangle$, where

$$\overline{b} = \frac{\langle f_1, f_2 \rangle}{a}$$

$$b = \frac{\langle f_2, f_1 \rangle}{\overline{a}}$$

We can calculate $b\overline{b}$ and $c\overline{c}$ to get

$$b\overline{b} = \frac{\langle f_1, f_2 \rangle \langle f_2, f_1 \rangle}{a\overline{a}} = \frac{|\langle f_1, f_2 \rangle|^2}{\|f_1\|^2} = \frac{1}{6}$$

and

$$c\overline{c} = \frac{1}{2} - b\overline{b} = \frac{1}{3}.$$

Thus

$$\mathbf{F} = \tilde{f}_1 \otimes \tilde{f}_1 + \tilde{f}_2 \otimes \tilde{f}_2 = \begin{bmatrix} a\bar{a} + b\bar{b} & b\bar{c} \\ \bar{b}c & c\bar{c} \end{bmatrix}.$$

To find the norm of the above matrix we solve the characteristic equation

$$(a\bar{a} + b\bar{b} - \lambda)(c\bar{c} - \lambda) - b\bar{b}c\bar{c} = 0.$$

Interestingly, we know all of the quantities and can therefore solve for λ. Hence,

$$\left(\frac{2}{3} - \lambda\right)\left(\frac{1}{3} - \lambda\right) - \frac{1}{18} = 0.$$

Which becomes,

$$\lambda^2 - \lambda + \frac{1}{6} = 0$$

The solutions to this equation are $\{\lambda_1, \lambda_2\} = \{\frac{1}{6}\left(3 - \sqrt{3}\right), \frac{1}{6}\left(3 + \sqrt{3}\right)\}$. The norm is equal to the largest eigenvalue; hence,

$$\|\mathbf{F}\| = \max\{\lambda_1, \lambda_2\} = \frac{1}{6}\left(3 + \sqrt{3}\right) = 0.788675\ldots$$

This norm is exactly the value from Table 3.1.

Suppose we consider the case where $S_1 = \{1, 2, 3\}$ where a third vector f_3 is added. We find the corresponding vector $\tilde{f}_3 = \mathbf{U}f_3 = \{d, e\}^T$. If we repeat a similar analysis as we did for f_1 we get that $d\bar{d} = \frac{1}{6}$ and $e\bar{e} = \frac{1}{2}$. Our sum of outer products is

$$\mathbf{F} = \tilde{f}_1 \otimes \tilde{f}_1 + \tilde{f}_2 \otimes \tilde{f}_2 + \tilde{f}_3 \otimes \tilde{f}_3 = \begin{bmatrix} a\overline{a} + b\overline{b} + d\overline{d} & b\overline{c} + d\overline{e} \\ \overline{b}c + \overline{d}e & c\overline{c} + e\overline{e} \end{bmatrix}.$$

The corresponding characteristic equation becomes

$$\left(\frac{2}{3} - \lambda\right)\left(\frac{5}{6} - \lambda\right) - \frac{1}{18} = 0,$$

which simplifies to

$$\lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = 0.$$

The solutions to this equation are $\{\lambda_1, \lambda_2\} = \{\frac{1}{2}, 1\}$. The norm is equal to the largest eigenvalue; hence,

$$\|\mathbf{F}\| = \max\{\lambda_1, \lambda_2\} = 1.$$

This norm is exactly the value from Table 3.1.

Case *n* = 8. Let us consider $n = 8$. This will yield eight f_k vectors in $\mathbb{C}^4$, where the $\mathbf{R}$- matrix is given by

$$\mathbf{R} = \begin{bmatrix} 1 & -\frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} \\ \frac{i}{\sqrt{7}} & 1 & \frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & \frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & \frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} \\ \frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & 1 & \frac{i}{\sqrt{7}} & \frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & \frac{i}{\sqrt{7}} \\ \frac{i}{\sqrt{7}} & \frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & 1 & \frac{i}{\sqrt{7}} & \frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} \\ \frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & 1 & \frac{i}{\sqrt{7}} & \frac{i}{\sqrt{7}} & \frac{i}{\sqrt{7}} \\ \frac{i}{\sqrt{7}} & \frac{i}{\sqrt{7}} & \frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & 1 & -\frac{i}{\sqrt{7}} & \frac{i}{\sqrt{7}} \\ \frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & \frac{i}{\sqrt{7}} & \frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & \frac{i}{\sqrt{7}} & 1 & -\frac{i}{\sqrt{7}} \\ \frac{i}{\sqrt{7}} & \frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & \frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & -\frac{i}{\sqrt{7}} & \frac{i}{\sqrt{7}} & 1 \end{bmatrix} \tag{3.25}$$

The corresponding $\mathbb{W}$-matrix is

$$\mathbb{W} = \begin{bmatrix} \frac{i\left(i+\sqrt{7}\right)}{2\sqrt{14}} & \frac{2i}{\sqrt{21}} & \frac{i\left(i+\sqrt{7}\right)}{2\sqrt{21}} & \frac{i}{\sqrt{14}} \\ \frac{i\left(i+\sqrt{7}\right)}{2\sqrt{14}} & \frac{7-3i\sqrt{7}}{14\sqrt{3}} & \frac{1+i\sqrt{7}}{2\sqrt{21}} & -\frac{i}{\sqrt{14}} \\ \frac{1-i\sqrt{7}}{2\sqrt{14}} & \frac{7+3i\sqrt{7}}{14\sqrt{3}} & \frac{1+i\sqrt{7}}{2\sqrt{21}} & -\frac{i}{\sqrt{14}} \\ \frac{3+i\sqrt{7}}{2\sqrt{14}} & \frac{i}{\sqrt{21}} & \frac{1-i\sqrt{7}}{2\sqrt{21}} & -\frac{i}{\sqrt{14}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \sqrt{\frac{3}{7}} & \frac{i}{\sqrt{14}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{i\left(-i+\sqrt{7}\right)}{2\sqrt{21}} & \frac{i}{\sqrt{14}} \\ \sqrt{\frac{2}{7}} & -\frac{i}{\sqrt{21}} & \frac{i\left(i+\sqrt{7}\right)}{2\sqrt{21}} & \frac{i}{\sqrt{14}} \end{bmatrix} \tag{3.26}$$

Each row in (3.26) is one frame vector f_k. Thus our frame $\mathcal{F} = \{f_1, ..., f_8\}$.

We now consider the problem of finding a partition $\{S_1, ..., S_r\}$ of $[8]$ such that $[8] = \{1, ..., 8\} = S_1 \cup \cdots \cup S_r$, where r is between 1 and 8. In order to examine the norms corresponding to the subsets of the partition, we need to calculate the norms of all the different possible combinations of the vectors. There are $2^n - 1 = 2^8 - 1 = 255$ possible combinations of picking a set of vectors out of $\mathcal{F}$. Table 4 shows the corresponding norms for the given combinations of vectors. As before, we observe that the norms only take on specific values.

Table 4

Subset size and the corresponding norms for n = 8 case

$\lvert S_k \rvert$	$\lVert \mathbf{F} \rVert$
1	0.5
2	0.688982
3	0.827327
4	0.827327 or 0.956243
≥ 5	1

Let us perform the same analysis on the number of subsets r as we did in the $n = 4$ case to estimate the bound in Corollary 1.5:

$$\begin{aligned}
1 \leq r \leq 8 &\Rightarrow 1 \leq \sqrt{r} \leq 2\sqrt{2} \\
&\Rightarrow 1 \geq \frac{1}{\sqrt{r}} \geq \frac{\sqrt{2}}{4} \\
&\Rightarrow 1 + \frac{1}{\sqrt{2}} \geq \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{2}} \geq \frac{\sqrt{2}}{4} + \frac{2\sqrt{2}}{4} \\
&\Rightarrow \left(1 + \frac{1}{\sqrt{2}}\right)^2 \geq \left(\frac{1}{\sqrt{r}} + \frac{1}{\sqrt{2}}\right)^2 \geq \left(\frac{3\sqrt{2}}{4}\right)^2 \\
&\Rightarrow \left(1 + \frac{1}{\sqrt{2}}\right)^2 \geq \left(\frac{1}{\sqrt{r}} + \frac{1}{\sqrt{2}}\right)^2 \geq \frac{9}{8} = 1.125 \qquad (3.27)
\end{aligned}$$

Notice that our corollary bound is again always greater than unity; hence, any partition of $\mathcal{F}$ will satisfy the corollary 1.5. Interestingly, our bound has decreased from $1.457106...$ for the $n = 4$ case to 1.125 for $n = 8$ case. This is to be expected because as r increases the corollary bound will approach 0.5. We need to examine larger frames to see if this continues to hold and we need to find a relationship between n and r.

Case *n* = 16. If we repeat the same procedure for the case of $n = 16$ we have that the entries of the 16×16 $\mathbf{R}$-matrix is given by

$$\mathbf{R}_{j,k} = \begin{cases} 1 & j = k; \\ \frac{\pm i}{\sqrt{15}} & j \neq k. \end{cases} \tag{3.28}$$

The corresponding $\mathbb{W}$-matrix with rows that are the frame vectors f_k is given by

$$
\mathbb{W} = \begin{bmatrix}
\frac{i}{4}-\frac{1}{4\sqrt{15}} & \frac{-15+7i\sqrt{15}}{30\sqrt{14}} & \frac{i\left(i+\sqrt{15}\right)}{2\sqrt{70}} & \frac{5+3i\sqrt{15}}{20\sqrt{3}} & \frac{i\left(i+\sqrt{15}\right)}{6\sqrt{10}} & \frac{4i}{3\sqrt{35}} & \frac{i\left(i+\sqrt{15}\right)}{2\sqrt{105}} & \frac{i}{\sqrt{30}} \\
\frac{i}{4}-\frac{1}{4\sqrt{15}} & -2i\sqrt{\frac{2}{105}} & \frac{1}{3}\sqrt{\frac{2}{35}}\left(1+i\sqrt{15}\right) & \frac{15-7i\sqrt{15}}{60\sqrt{3}} & \frac{i\left(i+\sqrt{15}\right)}{6\sqrt{10}} & \frac{15-7i\sqrt{15}}{30\sqrt{21}} & \frac{1+i\sqrt{15}}{2\sqrt{105}} & -\frac{i}{\sqrt{30}} \\
\frac{1}{60}\left(-15i+\sqrt{15}\right) & 2i\sqrt{\frac{2}{105}} & \frac{-11+3i\sqrt{15}}{6\sqrt{70}} & \frac{15-7i\sqrt{15}}{60\sqrt{3}} & \frac{1-i\sqrt{15}}{6\sqrt{10}} & \frac{15+7i\sqrt{15}}{30\sqrt{21}} & \frac{1+i\sqrt{15}}{2\sqrt{105}} & -\frac{i}{\sqrt{30}} \\
\frac{i}{4}-\frac{1}{4\sqrt{15}} & \frac{-15+7i\sqrt{15}}{30\sqrt{14}} & \frac{1}{3}\sqrt{\frac{2}{35}}\left(1-i\sqrt{15}\right) & \frac{15-7i\sqrt{15}}{60\sqrt{3}} & \frac{3+i\sqrt{15}}{6\sqrt{10}} & \frac{i}{\sqrt{35}} & \frac{1-i\sqrt{15}}{2\sqrt{105}} & -\frac{i}{\sqrt{30}} \\
\frac{1}{60}\left(-15i+\sqrt{15}\right) & \frac{15-7i\sqrt{15}}{30\sqrt{14}} & \frac{1-i\sqrt{15}}{2\sqrt{70}} & \frac{5+i\sqrt{15}}{10\sqrt{3}} & \frac{1+i\sqrt{15}}{6\sqrt{10}} & \frac{15+7i\sqrt{15}}{30\sqrt{21}} & \frac{1+i\sqrt{15}}{2\sqrt{105}} & -\frac{i}{\sqrt{30}} \\
\frac{1}{60}\left(-15i+\sqrt{15}\right) & 2i\sqrt{\frac{2}{105}} & \frac{17+3i\sqrt{15}}{6\sqrt{70}} & \frac{i}{3\sqrt{5}} & \frac{1+i\sqrt{15}}{6\sqrt{10}} & -\frac{4i}{3\sqrt{35}} & \frac{1-i\sqrt{15}}{2\sqrt{105}} & -\frac{i}{\sqrt{30}} \\
\frac{i}{4}-\frac{1}{4\sqrt{15}} & \frac{45+7i\sqrt{15}}{30\sqrt{14}} & \frac{1-i\sqrt{15}}{3\sqrt{70}} & \frac{i}{3\sqrt{5}} & -\frac{i\left(-i+\sqrt{15}\right)}{6\sqrt{10}} & \frac{15-7i\sqrt{15}}{30\sqrt{21}} & \frac{1+i\sqrt{15}}{2\sqrt{105}} & -\frac{i}{\sqrt{30}} \\
\frac{i}{4}+\frac{7}{4\sqrt{15}} & -i\sqrt{\frac{2}{105}} & \frac{1+i\sqrt{15}}{3\sqrt{70}} & \frac{i}{3\sqrt{5}} & \frac{3-i\sqrt{15}}{6\sqrt{10}} & \frac{i}{\sqrt{35}} & \frac{1-i\sqrt{15}}{2\sqrt{105}} & -\frac{i}{\sqrt{30}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{7}{15}} & \frac{i}{\sqrt{30}} \\
0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{7}} & -\frac{i\left(-i+\sqrt{15}\right)}{2\sqrt{105}} & \frac{i}{\sqrt{30}} \\
0 & 0 & 0 & 0 & \sqrt{\frac{2}{5}} & -\frac{i}{\sqrt{35}} & \frac{i\left(i+\sqrt{15}\right)}{2\sqrt{105}} & \frac{i}{\sqrt{30}} \\
0 & 0 & 0 & \frac{1}{\sqrt{3}} & -\frac{i\left(-i+\sqrt{15}\right)}{6\sqrt{10}} & \frac{-15-7i\sqrt{15}}{30\sqrt{21}} & -\frac{i\left(-i+\sqrt{15}\right)}{2\sqrt{105}} & \frac{i}{\sqrt{30}} \\
0 & 0 & \frac{\sqrt{\frac{14}{5}}}{3} & -\frac{i}{3\sqrt{5}} & -\frac{i\left(-i+\sqrt{15}\right)}{6\sqrt{10}} & \frac{4i}{3\sqrt{35}} & \frac{i\left(i+\sqrt{15}\right)}{2\sqrt{105}} & \frac{i}{\sqrt{30}} \\
0 & \sqrt{\frac{2}{7}} & \frac{i\left(i+\sqrt{15}\right)}{3\sqrt{70}} & -\frac{i}{3\sqrt{5}} & \frac{1+i\sqrt{15}}{6\sqrt{10}} & \frac{-15+7i\sqrt{15}}{30\sqrt{21}} & -\frac{i\left(-i+\sqrt{15}\right)}{2\sqrt{105}} & \frac{i}{\sqrt{30}} \\
\frac{2}{\sqrt{15}} & i\sqrt{\frac{2}{105}} & -\frac{i\left(-i+\sqrt{15}\right)}{3\sqrt{70}} & -\frac{i}{3\sqrt{5}} & \frac{i\left(3i+\sqrt{15}\right)}{6\sqrt{10}} & -\frac{i}{\sqrt{35}} & \frac{i\left(i+\sqrt{15}\right)}{2\sqrt{105}} & \frac{i}{\sqrt{30}}
\end{bmatrix}
$$

There are $2^{16}-1$ possible subsets of $\mathcal{F}$. Table 5 shows the norms corresponding to a subset S_k with cardinality k.

Table 5

Subset size and the corresponding norms for n = 16 case.

$\lvert S_k \rvert$	$\lVert \mathbf{F} \rVert$
1	0.5
2	0.629099
3	0.723607
4	0.723607 or 0.811674
5	0.841565 or 0.897327
6	0.841565 or 0.887298 or 0.897327 or 0.919462 or 0.942428
7	0.841565 or 0.928174 or 0.95929 or ... or 1
8	0.841565 or 0.95929 or 0.971249 or 0.977324 or 0.981806 or 0.995257 or 1
≥ 9	1

Two noticeable observations for $n = 16$:

- As we increase the number of elements in a subset the number of values, not equal to 1, that the corresponding norm takes on increases. For instance, a subset with 4 elements, $|S_k| = 4$, corresponded with two possible norms not equal to 1. If a subset contained 6 elements, $|S_k| = 6$, this corresponded to five possible norms not equal to 1.
- If the cardinality of a subset is greater than 9, the corresponding norm is always 1. Nine is one more than half the cardinality of $\mathcal{F}$.

If we analyze the range of values on r we have,

$$\begin{aligned}
1 \leq r \leq 16 \Rightarrow 1 \leq \sqrt{r} \leq 4 \\
&\Rightarrow 1 \geq \frac{1}{\sqrt{r}} \geq \frac{1}{4} \\
&\Rightarrow 1 + \frac{1}{\sqrt{2}} \geq \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{2}} \geq \frac{1}{4} + \frac{1}{\sqrt{2}} \\
&\Rightarrow 1 + \frac{1}{\sqrt{2}} \geq \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{2}} \geq \frac{1+2\sqrt{2}}{4} \\
&\Rightarrow \left(1 + \frac{1}{\sqrt{2}}\right)^2 \geq \left(\frac{1}{\sqrt{r}} + \frac{1}{\sqrt{2}}\right)^2 \geq \left(\frac{1+2\sqrt{2}}{4}\right)^2 \\
&= \frac{9+4\sqrt{2}}{16} = 0.916053...
\end{aligned}$$

Notice that the corollary bound falls below unity; therefore, the lowest that our corollary bound may achieve occurs when $r = 16$ and the bound is $0.916053...$ The question that remains is whether there exists a subset S_k with corresponding norm that is greater than the corollary bound. Keep in mind that the size of the largest subset will dictate the number of r partitions that can be formed from $[16] = \{1, ..., 16\}$. For instance, suppose $|S_1| = 8$. There are only eight remaining vectors left in our frame to distribute among the other subsets; therefore, the largest number of subsets that can be formed, with one subset of size equal to eight, is $r = 9$. This means the set of cardinalities for the nine subsets is $\{|S_k|\}_{k=1}^{9} = \{8, 1, 1, 1, 1, 1, 1, 1, 1\}$. In Table 6, which shows the corollary bounds for all the possible r subsets, we see that that the bound for $r = 9$ is 1.08252. Since all the corresponding norms in Table 5 fall below 1.08252, this shows that $\{S_k\}_{k=1}^{9}$

is a valid partition.

Table 6

Corollary bounds for different r's ($n = 16$ case)

r	Cor. Bnd. (1.2)
1	2.91421...
2	2
3	1.64983...
4	1.45711...
5	1.33246...
6	1.24402...
7	1.17738...
8	1.125
9	1.08252...
10	1.04721...
11	1.01731...
12	0.991582...
13	0.969155...
14	0.949393...
15	0.931815...
16	0.916053...

As another example, suppose we stumble upon a partition and the largest subset corresponds to 5 of our frame vectors from $\mathcal{F}$. Therefore, the smallest number of subsets we can have in our partition is $r = 4$, e.g., $\{|S_k|\}_{k=1}^{4} = \{5, 5, 5, 1\}$. The largest number of subsets we can have in our partition is $r = 12$, e.g., $\{|S_k|\}_{k=1}^{12} = \{5, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$. Hence for a partition whose largest subset has cardinality 5, for $4 \leq r \leq 12$. This in turn corresponds to the corollary bound falling between $0.991582...$ and $1.45711...$, but the norms corresponding to the subsets for r in this range will at most be 0.897327 as shown in Table 6. This means any partition whose largest subset cardinality 5 will be a valid partition.

Table 7

Max Sized Partition, Corollary Bounds, and Norm

$\{\lvert S_k\rvert\}_{max}$	$r \in [...]$	Cor. Bound Range	Max Norm
1	$r = 16$	exactly $0.916053...$	0.5
2	$[8...15]$	$[0.931815, 1.125]$	0.629099
3	$[6...14]$	$[0.949393, 1.24402]$	0.723607
4	$[4...13]$	$[0.969155, 1.45711]$	0.811674
5	$[4...12]$	$[0.991582, 1.45711]$	0.897327
6	$[3...11]$	$[1.01731, 1.64983]$	0.942428
7	$[3...10]$	$[1.04721, 1.64983]$	1
8	$[2...9]$	$[1.08252, 2]$	1
9	$[2...8]$	$[1.125, 2]$	1
10	$[2...7]$	$[1.17738, 2]$	1
11	$[2...6]$	$[1.24402, 2]$	1
12	$[2...5]$	$[1.33246, 2]$	1
13	$[2...4]$	$[1.45711, 2]$	1
14	$r = \{2, 3\}$	$\{1.64983, 2\}$	1
15	$r = 2$	$\{2\}$	1
16	$r = 1$	$\{2.91421\}$	1

Table 7 summarizes our results, which shows all partitions are valid. To explain the table, the first column gives the range of values for the largest subset in a partition. The second column gives the range of values for r. The third column indicates the range of values for the corollary bound corresponding to the range of for r. The fourth column gives the norm corresponding to the largest subset. Notice that no matter what the cardinalitity of the largest subset is, the norm in the fourth column is always less than the lower bound in the third column; hence any partition of the 16 frame vectors is a valid partition.

Case *n* = 32. We can follow the same construction as in the previous sections by defining our $\mathbf{R}$-matrix to be

$$\mathbf{R}_{j,k} = \begin{cases} 1 & j = k; \\ \pm \frac{i}{\sqrt{31}} & j \neq k. \end{cases} \tag{3.29}$$

The corresponding $\mathbb{W}$-matrix has 32 rows, corresponding to the 32 frame vectors, and 16 columns. Previously, we concluded for the cases where $n = 4, 8, 16$ that any partition of the frame is valid. However for $n = 32$ we will show that this is not case. Therefore, the following questions need to be explored:

- Since we have 32 frame vectors to work with will any partition of $[32]$ be a valid partition?
- For $n = 32$, does it matter if the vectors are *consecutive* or *non-consecutive*?

Let us workout an example. Suppose we let $r = 17$. We need to partition $[32]$ into 17 subsets S_k such that

$$[32] = S_1 \cup S_2 \cup ... \cup S_{17},$$

where

$$32 = \sum_{k=1}^{17} |S_k|.$$

For example consider the partition $\{S_1, \dots, S_{17}\}$ whose cardinalities are given by

$$P_1 = \{|S_k|\}_{k=1}^{17} = \{16, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}.$$

The corollary bound for $r = 17$ is,

$$\left(\frac{1}{\sqrt{17}} + \frac{1}{\sqrt{2}}\right)^2 = 0.901821... \tag{3.30}$$

The first column in Table 8 shows the norms corresponding to the partition whose cardinalities are given by P_1. The second and third columns correspond to partitions whose cardinalities are given by

$$P_2 = \{3, 3, 3, 3, 3, 3, 3, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1\},$$
$$P_3 = \{2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1\},$$

respectively. Observe that in each of these partitions the subsets consist of consecutive vectors in $\mathcal{F}$.

Table 8

Norms for partitions P_1, P_2, P_2 (Consecutive)

	P_1	Norm	P_2	Norm	P_3	Norm
$S_1 =$	$\{1...16\}$	0.847804...	$\{1,2,3\}$	0.655543...	$\{1,2\}$	0.589803...
$S_2 =$	$\{17\}$	0.5	$\{4,5,6\}$	0.655543...	$\{3,4\}$	0.589803...
$S_3 =$	$\{18\}$	0.5	$\{7,8,9\}$	0.655543...	$\{5,6\}$	0.589803...
$S_4 =$	$\{19\}$	0.5	$\{10,11,12\}$	0.655543...	$\{7,8\}$	0.589803...
$S_5 =$	$\{20\}$	0.5	$\{13,14,15\}$	0.655543...	$\{9,10\}$	0.589803...
$S_6 =$	$\{21\}$	0.5	$\{16,17,18\}$	0.655543...	$\{11,12\}$	0.589803...
$S_7 =$	$\{22\}$	0.5	$\{19,20,21\}$	0.655543...	$\{13,14\}$	0.589803...
$S_8 =$	$\{23\}$	0.5	$\{22,23\}$	0.589803...	$\{15,16\}$	0.589803...
$S_9 =$	$\{24\}$	0.5	$\{24\}$	0.5	$\{17,18\}$	0.589803...
$S_{10} =$	$\{25\}$	0.5	$\{25\}$	0.5	$\{19,20\}$	0.589803...
$S_{11} =$	$\{26\}$	0.5	$\{26\}$	0.5	$\{21,22\}$	0.589803...
$S_{12} =$	$\{27\}$	0.5	$\{27\}$	0.5	$\{23,24\}$	0.589803...
$S_{13} =$	$\{28\}$	0.5	$\{28\}$	0.5	$\{25,26\}$	0.589803...
$S_{14} =$	$\{29\}$	0.5	$\{29\}$	0.5	$\{27,28\}$	0.589803...
$S_{15} =$	$\{30\}$	0.5	$\{30\}$	0.5	$\{29,30\}$	0.589803...
$S_{16} =$	$\{31\}$	0.5	$\{31\}$	0.5	$\{32\}$	0.5
$S_{17} =$	$\{32\}$	0.5	$\{32\}$	0.5	$\{32\}$	0.5

Moreover Table 8 shows that the norms corresponding to each subsets is always less than the corollary bound (3.30). In fact, upon further computation it appears that when consecutive vectors are as subsets, the partition is valid. But what happens when our subsets do not consist of consecutive vectors? We consider three such examples of partitions in Table 9 where the cardinalities are again given by P_1, P_2, and P_3 but the subsets do not necessarily consist of consecutive vectors. For P_1, notice that every norm is the same except for the very first subset, whose norm exceeds the corollary bound. Thus the corresponding partition is *not* valid.

Conclusion

We set out to find frame partitioning algorithms and found that the type of frame affects the types of partitions we may form. Our research has demonstrated that for equinorm-equiangular frames we can always form an even split and almost even split partitions which are valid for the MSS Theorem. For partitions with unequal subsets there is a restriction on how we form the subsets; that is, the largest subset must be less than or equal to an n_{max}.

For Grassmannian frames we have shown that for orders $n = 4, 8, 16$ any partition is a valid partition. When we examine a Grassmannian frame for $n = 32$ we encounter examples of non-valid partitions. Based on the counterexample at the end of chapter 3 we conjecture the following:

- Partitions whose subsets consist of consecutive vectors are valid.
- For partitions whose subsets consist of non-consecutive vectors, and whose largest subset has cardinality at least half the cardinality of the frame, are not valid.

We believe that a proof of these conjectures will involve a good understanding of the frame correlation described at the beginning of the chapter. We now discuss open problems for future research:

- How can we partition Gabor frames under the same restrictions stated by the MSS Theorem?

- How can we easily calculate the norms of very large matrices formed from Grassmannian frames?

- For equinorm-equiangular frames we only considered partitions involving two subsets. How can we find partitions with three or more subsets?

- Another question to explore is: how do we choose vectors to be in a subset? For this research we considered consecutive vectors in a subset. What are the results if we consider odd or even vectors in a subset?

- Are there *efficient* ways of partitioning the frames we have studied so far?

These are questions that we have encountered in our analysis of equinorm-equiangular and Grassmannian frames. If we can understand and answer these questions we may be able to find more ways to partition sets of vectors.

Table 9

Norms for partitions P_1, P_2, P_3 (Non-Consecutive)

	P_1	Norm	P_2	Norm	P_3	Norm
$S_1 =$	$\{2, 5, 7, 8, 9, 10, 11, 13, 14, 16, 17, 19, 21, 25, 27, 31\}$	0.997832...	$\{10, 13, 30\}$	0.655543...	$\{3, 9\}$	0.589803...
$S_2 =$	$\{12\}$	0.5	$\{24, 25, 29\}$	0.655543...	$\{20, 23\}$	0.589803...
$S_3 =$	$\{22\}$	0.5	$\{5, 17, 21\}$	0.655543...	$\{5, 13\}$	0.589803...
$S_4 =$	$\{4\}$	0.5	$\{8, 12, 31\}$	0.655543...	$\{16, 18\}$	0.589803...
$S_5 =$	$\{23\}$	0.5	$\{3, 23, 32\}$	0.655543...	$\{11, 25\}$	0.589803...
$S_6 =$	$\{32\}$	0.5	$\{7, 14, 28\}$	0.655543...	$\{8, 31\}$	0.589803...
$S_7 =$	$\{3\}$	0.5	$\{1, 2, 11\}$	0.655543...	$\{24, 26\}$	0.589803...
$S_8 =$	$\{1\}$	0.5	$\{4, 9\}$	0.589803...	$\{7, 28\}$	0.589803...
$S_9 =$	$\{18\}$	0.5	$\{6\}$	0.5	$\{14, 30\}$	0.589803...
$S_{10} =$	$\{30\}$	0.5	$\{16\}$	0.5	$\{10, 27\}$	0.589803...
$S_{11} =$	$\{6\}$	0.5	$\{22\}$	0.5	$\{15, 19\}$	0.589803...
$S_{12} =$	$\{15\}$	0.5	$\{27\}$	0.5	$\{22, 29\}$	0.589803...
$S_{13} =$	$\{20\}$	0.5	$\{20\}$	0.5	$\{6, 21\}$	0.589803...
$S_{14} =$	$\{26\}$	0.5	$\{18\}$	0.5	$\{1, 4\}$	0.589803...
$S_{15} =$	$\{28\}$	0.5	$\{26\}$	0.5	$\{2, 32\}$	0.589803...
$S_{16} =$	$\{24\}$	0.5	$\{19\}$	0.5	$\{12\}$	0.5
$S_{17} =$	$\{29\}$	0.5	$\{15\}$	0.5	$\{17\}$	0.5

References

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Appendix A

Matrix and Vector Norm Proofs

A.1 Norm of the Outer-Product of a vector

Suppose $\|u\| = a$. Then what is the quantity $\|uu^*\|$? First notice that $\|u\|^2 = u^*u = a^2$ and let **U**$= uu^*$. With the definition of a matrix norm, we get

$$\|\mathbf{U}x\|^2 = (\mathbf{U}x)^*(\mathbf{U}x)$$

$$\begin{aligned}
&= x^*\mathbf{U}^*\mathbf{U}x \\
&= x^*(uu^*)^*uu^*x \\
&= x^*uu^*uu^*x \\
&= x^*u(u^*u)u^*x \\
&= x^*ua^2u^*x \\
&= a^2x^*uu^*x \\
&= a^2x^*\mathbf{U}x
\end{aligned} \tag{A.1}$$

Now we will utilize Lagrange multipliers to find the eigenvalue λ_{max}. Let

$$f(x, x^*) = a^2x^*\mathbf{U}x$$

and

$$g(x, x^*) = x^*x - 1 = 0.$$

Recall that the norm of a matrix is over unit vectors x, i.e., $x^*x = \|x\|^2 = 1$. Now define,

$$\begin{aligned} h(x, x^*, \lambda) &= f(x, x^*) - \lambda g(x, x^*) \\ &= a^2 x^* \mathbf{U} x - \lambda x^* x + \lambda \end{aligned}$$

Next differentiate the function h with respect to x to get

$$\frac{\partial h}{\partial x} = a^2 x^* \mathbf{U} - \lambda x^* = 0$$

We have set the derivative equal to 0 because our goal is to maximize the norm of the product of the matrix with every possible unit vector in $\mathbb{C}^{m \times n}$. Hence

$$a^2 x^* \mathbf{U} = \lambda x^* \tag{A.2}$$

Substitute (A.2) into (A.1) to get

$$\|\mathbf{U}x\|^2 = \lambda x^* x = \lambda \tag{A.3}$$

But the question that remains is what is the value of λ? Let us examine (A.2) closely:

$$\begin{aligned} a^2 x^* \mathbf{U} = a^2 x^* u u^* = \lambda x^* &\Rightarrow a^2 x^* u u^* u = \lambda x^* u \\ &\Rightarrow a^4 x^* u = \lambda x^* u \\ &\Rightarrow a^4 = \lambda \end{aligned}$$

Therefore (A.3) becomes

$$\|\mathbf{U}x\| = \sqrt{\lambda} = a^2$$

and the norm $\|uu^*\| = a^2 = \|u\|^2$. Therefore,

$$\|uu^*\| = \|u\|^2 \tag{A.4}$$

This also leads to the following corollary:

Corollary A.11 For any unit vector $u \in \mathbb{C}^d$ we have

$$\|uu^*\| = 1 \tag{A.5}$$

A.2 Analysis of Norm of the Sum of Outer Product of Two Different vectors

Suppose we have two vectors in $\mathbb{C}^m$

$$u = \{u_1, u_2, ..., u_m\}$$

$$v = \{v_1, v_2, ..., v_m\}$$

Assume $\|u\| = a$ and $\|v\| = b$. We wish to calculate $\|uu^* + vv^*\|$. Let

$$\mathbf{U} = uu^* \tag{A.6}$$

$$\mathbf{V} = vv^* \tag{A.7}$$

Notice that both matrices are Hermitian because $\mathbf{U}^* = (uu^*) = uu^* = \mathbf{U} \Rightarrow \mathbf{U}^* = \mathbf{U}$. Since both matrices $\mathbf{U}$ and $\mathbf{V}$ are Hermitian, the sum of both matrices is also Hermitian:

$$\begin{aligned} \mathbf{M} = \mathbf{U} + \mathbf{V} \Rightarrow \mathbf{M}^* &= (\mathbf{U} + \mathbf{V})^* \\ &= \mathbf{U}^* + \mathbf{V}^* \\ &= \mathbf{U} + \mathbf{V} = \mathbf{M} \end{aligned}$$

Therefore, $\mathbf{M}^* = \mathbf{M}$. If we wish to find the norm of the sum $\mathbf{M} = \mathbf{U} + \mathbf{V}$, then we need to find a solution to $\|\mathbf{M}x\|^2$. Hence,

$$\begin{aligned} \|\mathbf{M}x\|^2 &= (\mathbf{M}x)^* \mathbf{M}x \\ &= x^* \mathbf{M}^* \mathbf{M} x \\ &= x^* \mathbf{M}^2 x \\ &= x^* (\mathbf{U} + \mathbf{V}) (\mathbf{U} + \mathbf{V}) x \\ &= x^* \left(\mathbf{U}^2 + \mathbf{U}\mathbf{V} + \mathbf{V}\mathbf{U} + \mathbf{V}^2 \right) x \\ &= x^* (uu^*uu^* + uu^*vv^* + vv^*uu^* + vv^*vv^*) x \\ &= x^* (auu^* + uu^*vv^* + vv^*uu^* + bvv^*) x \end{aligned} \tag{A.8}$$

Or in a simpler form,

$$= x^* \left(a\mathbf{U} + \mathbf{U}\mathbf{V} + \mathbf{V}\mathbf{U} + b\mathbf{V} \right) x$$

$$= ax^*\mathbf{U}x + x^*\mathbf{U}\mathbf{V}x + x^*\mathbf{V}\mathbf{U}x + bx^*\mathbf{V}x \tag{A.9}$$

Notice that if our two vectors u and v are orthogonal, then (A.8) reduces to

$$\|\mathbf{M}x\|^2 = x^* \left(a\mathbf{U} + b\mathbf{V} \right) x \tag{A.10}$$

Appendix B

General Formulas

B.1 Summation Formulas

The following formulas are utilized from [7]:

$$\sum_{j=0}^{m-1} \sin jy = \sin\left(\frac{m-1}{2}y\right)\sin\left(\frac{my}{2}\right)\frac{1}{\sin\left(\frac{y}{2}\right)} \tag{B.1}$$

$$\sum_{j=0}^{m-1} \cos jy = \cos\left(\frac{m-1}{2}y\right)\sin\left(\frac{my}{2}\right)\frac{1}{\sin\left(\frac{y}{2}\right)} \tag{B.2}$$

Let $y = \frac{2\pi}{m}$. We get

$$\sum_{j=0}^{m-1} \sin\left(\frac{2\pi j}{m}\right) = \sin\left(\frac{m-1}{m}\pi\right)\sin(\pi)\frac{1}{\sin\left(\frac{\pi}{m}\right)} = 0 \tag{B.3}$$

and

$$\sum_{j=0}^{m-1} \cos\left(\frac{2\pi j}{m}\right) = \cos\left(\frac{m-1}{m}\pi\right)\sin(\pi)\frac{1}{\sin\left(\frac{\pi}{m}\right)} = 0. \tag{B.4}$$