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An analysis of spacetime numbers

Nicolae Andrew Borota
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AN ANALYSIS OF SPACETIME NUMBERS

by

Nicolae Andrew Borota

A Thesis

Submitted in partial fulfillment of the requirements of the Masters of Arts Degree of The Graduate School at Rowan University 04/30/99

Approved by ____________________________

Date Approved APRIL 30, 1999
The familiar complex numbers begin by considering the solution to the equation $i^2 = -1$, which is not a real number. A two-dimensional number system arises of the form $z = x + iy$. Spacetime numbers are based upon the simple relation $j^2 = 1$, with the corresponding two-dimensional number system $z = x + jt$. It seems odd that anything useful can come from this simple idea, since the solutions of our $j$ equation are the familiar real numbers $+1$ and $-1$. However, many interesting applications arise from these new numbers. The most useful aspect of spacetime numbers is in solving problems in the areas of special and general relativity. These areas deal with the notion of space-time, hence the name "spacetime numbers." My goal is to explain in a direct, yet simple manner, the use of these special numbers. I begin by comparing spacetime numbers to the more familiar complex numbers, then introduce the spacetime plane as a mathematical construction and show unusual features of spacetime arithmetic. A spacetime version of Euler's formula is then presented and then the solutions to the one-dimensional wave equation.
MINI-ABSTRACT

Nicolae Andrew Borota
An Analysis of Spacetime Numbers
1999
Thesis
Mathematics

The purpose of this thesis was to research how the spacetime number system and the complex number system relate to one another and analyze their striking differences. Beneath the simplistic equation that generates the spacetime number system, there lies a rich mathematical region that has applications to physics.
AN ANALYSIS OF SPACETIME NUMBERS

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Nicolae Andrew Borota
Acknowledgements

This thesis is dedicated first and foremost to my parents, Nicolae and Helen, without whose love and devotion none of this would have been possible. They allowed me quiet time and let the dog out whenever she was whining so as not to disturb my work in progress. I couldn’t have asked for more understanding parents. Many thanks also need to be extended to Dr. Thomas J. Osler and Dr. Eduardo Flores. They were the individuals who introduced me to the topic of spacetime numbers. Dr. Osler has always offered me words of encouragement and proofread my papers whenever I asked him to. He unselfishly shared his wealth of knowledge with me whenever I needed to be shown the correct path to take to arrive at the answer. Dr. Osler kindly donated many of the drawings used in this thesis. I would also like to say a word thanks to everyone in the Math Department at Rowan University. My professors were always there whenever I needed them. It has brought me great pleasure to be awarded the opportunity to work on something that I enjoy so much. I hope that I have made everyone proud.
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Chapter 1 Spacetime Arithmetic

1. Introduction

The subject of spacetime numbers, better known as hyperbolic numbers, is not new, but is not well known because it is not discussed in standard textbooks. Spacetime numbers are a special case of the more general Clifford Algebras [16]. Many books and articles on this subject begin by explaining that more students of mathematics and physics would profit from a rigorous mathematical format. These rigorous presentations, full of mathematical jargon, present an unnecessary barrier to many readers not so mathematically sophisticated. Many students soon tire of the massive new vocabulary placed before them by mathematical researchers and continue to use traditional mathematical tools like the complex numbers instead. This is unfortunate, because many of the features of these numbers can be understood by precalculus students. Spacetime numbers can be used with profit to solve problems in physics, especially in relativity theory [2, 4]. They also provide a nifty example for students in a first course in modern abstract algebra [7].

It is our purpose here to present the subject of spacetime numbers in a gentle and fun manner. We will avoid a rigorous definition, lemma, and theorem approach in favor of a much lighter, intuitive, self-discovery presentation. We invite the reader to join us on a pleasant exploration of a new and useful idea, and leave it to the references to fill in the needed mathematical rigor. Spacetime numbers behave in
many ways like the familiar complex numbers, and we will use this analogy to help us understand them. On the other hand, spacetime numbers at times have surprising features never seen before by many students. These spicy new ideas should keep our adventure lively.

So let us begin. Just what are spacetime numbers all about...

2. **Complex Numbers vs. Spacetime Numbers.**

In high school algebra courses, the complex numbers are often introduced in a simple natural way. The easy steps are:

(a) Argue that no number familiar to the students satisfies $i = \sqrt{-1}$.

(b) Argue that $i^2 = -1, i^3 = -i, i^4 = 1, \ldots$

(c) Introduce the quantity $x + iy$, where $x$ and $y$ are real numbers, as a new kind of number, call it a complex number.

(d) The arithmetic of these new numbers is invented using a simple principal. Manipulate using all the familiar rules of algebra, treating $i$ as you would any algebraic variable with the exception that whenever the quantity $i^2$ occurs, we replace it with $-1$.

Using these four easy steps, the arithmetic of the complex number system becomes assessable to precalculus students. In a similar way, we can introduce the less familiar spacetime numbers.
(a') We begin with the surprising simple relation \( j^2 = 1 \), which has the solution \( j = \pm 1 \). Unlike (a) above, these are not new numbers. Nevertheless, we will continue using \( j \) as though it were new and not replacing it by the numbers \( \pm 1 \).

(b') The list of powers of \( j \) is even simpler than the list of powers of \( i \):
\[
j^0 = 1, j^1 = \pm 1, j^2 = 1, j^3 = \pm 1, \cdots.
\]
The \( j \)'s form a two cycle, while the powers of \( i \) form a four cycle.

(c') Introduce the quantity \( x + j t \), where \( x \) is the space part and \( t \) is the time part, \( x \) and \( t \) are motivated by applications presented later.

(d')

<table>
<thead>
<tr>
<th></th>
<th>Complex Arithmetic</th>
<th>Spacetime Arithmetic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numbers</td>
<td>( Z = X + iY, z = x + iy )</td>
<td>( Z = X + jT, z = x + jt )</td>
</tr>
<tr>
<td>Addition</td>
<td>( Z + z = (X + x) + i(Y + y) )</td>
<td>( Z + z = (X + x) + j(T + t) )</td>
</tr>
<tr>
<td>Subtraction</td>
<td>( Z - z = (X - x) + i(Y - y) )</td>
<td>( Z - z = (X - x) + j(T - t) )</td>
</tr>
<tr>
<td>Multiplication</td>
<td>( Zz = (Xx - Yy) + i(Yx + yX) )</td>
<td>( Zz = (Xx + Tt) + j(Tx + tX) )</td>
</tr>
</tbody>
</table>
| Division       | \[
\frac{Z}{z} = \frac{X + iY}{x + iy} \cdot \frac{x - iy}{x - iy} = \frac{(Xx + Yy) + i(Yx - Xy)}{(x^2 + y^2)}
\] | \[
\frac{Z}{z} = \frac{X + jT}{x + jt} \cdot \frac{x - jt}{x - jt} = \frac{(Xx - Tt) + j(Tx - Xt)}{(x^2 - t^2)}
\] |

Table 1: Complex vs. Spacetime Arithmetic

The reader should verify the above four operations on spacetime numbers. We repeat that this is easily done by treating \( j \) as an ordinary algebraic variable and replacing \( j^2 \) by
1.

**Reflections:**

It is interesting to point out that at any time we can replace $j$ by $+1$ or $-1$ in any of the equations that we generate and the results would be correct. This is in contrast to the complex number system because there exists no such substitution for $i$. We chose to overlook these obvious solutions of $j^2 = 1$ and followed the steps we used in creating the complex number system (steps (a)-(d)). By doing this, we sought to produce a system resembling the complex number system, but instead of being defined by $i^2 = -1$, it was defined by $j^2 = 1$. There will be times when it will be to our advantage to replace $j$ by $+1$ and $-1$ and we will make the reader aware when the substitutions should be carried out.

3. **The Spacetime Plane and The Three Types of Spacetime Numbers.**

You construct the spacetime plane by labeling the horizontal axis as the space axis and the vertical axis as our time axis, so any spacetime number can be found in the plane.

When we divided two spacetime numbers, we saw that after multiplying through by the conjugate of the denominator, the result was $x^2 - t^2$. This quantity puts a stipulation on the division, for if $x^2 - t^2 = 0$ (i.e. $x^2 = t^2$), then the denominator would be equal to zero, which we know from ordinary arithmetic, is not allowed.

If we examine $x^2 - t^2$, we see that there are only three possibilities or cases.
Case I: $x^2 - t^2 = 0 \quad x^2 = t^2$ Light-like

Case II: $x^2 - t^2 > 0 \quad x^2 > t^2$ Space-like

Case III: $x^2 - t^2 < 0 \quad x^2 < t^2$ Time-like

When $x^2 = t^2$, we can solve to obtain $x = t$ or $x = -t$. If we graph these two lines $x = t$ and $x = -t$ in the spacetime plane (see Figure 1), we obtain what we will refer to as light-like lines. Any spacetime numbers lying on these lines will be referred to as light-like numbers.

![Figure 1: Light-like Numbers](image)

Now we examine Case II when $x^2 > t^2$. This means that the square of the space part of the number is greater that the square of the time part. This corresponds to the shaded areas in Figure 2. We refer to these areas as the space-like regions and thus, any number in these regions are space-like numbers.
Examining Case III, when $x^2 < t^2$, we see that this means that the square of the space part of the number is less than the square of the time part. This corresponds to the shaded areas in Figure 3. We refer to these areas as the time-like regions and thus, any number in these regions are time-like numbers.

4. An Unexpected Feature of Multiplication.

Now after observing that there are only three possible classes of spacetime numbers (light-like, space-like, time-like), we can construct a table to see what the
result would be when two light-like numbers are multiplied together, or when a space-like number and a time-like number are multiplied. The results are summarized in the following multiplication table.

<table>
<thead>
<tr>
<th>Multiplication</th>
<th>Light-like</th>
<th>Space-like</th>
<th>Time-like</th>
</tr>
</thead>
<tbody>
<tr>
<td>Light-like</td>
<td>Light-like</td>
<td>Light-like</td>
<td>Light-like</td>
</tr>
<tr>
<td>Space-like</td>
<td>Light-like</td>
<td>Space-like</td>
<td>Time-like</td>
</tr>
<tr>
<td>Time-like</td>
<td>Light-like</td>
<td>Time-like</td>
<td>Space-like</td>
</tr>
</tbody>
</table>

Table 2: Spacetime Multiplication

We can verify any of these entries. Let's take an example of multiplying a light-like number and a time-like number. Remember that a light-like number is one in which $x^2 = t^2$, and a time-like number is one in which $x^2 < t^2$. We arbitrarily choose our light-like number to be $z_1 = -3 + 3j$ and our time-like number to be $z_2 = 4 + 6j$. Now we must perform the multiplication.

$$z \times z = (-3 + 3j)(4 + 6j)$$

$$= (-3 \times 4) + (-3 \times 6j) + (3j \times 4) + (3j \times 6j)$$

$$= (-12) + (-18j) + (12j) + (18j^2)$$

$$= (-12 + 18) + (-18j + 12j)$$

$$= 6 - 6j$$

Which is indeed a light-like number because $x = 6$ and $t = -6$ so $x^2 = 6^2 = 36$ and $t^2 = (-6)^2 = 36$, thus $x^2 = t^2$. The reader can proceed at his/her convenience to verify the other entries in the table if they are so inclined.

The multiplication table we have generated resembles the multiplication of real
numbers. If we take any real number(s) there are three possibilities, the number can be zero, positive, or negative, so we can construct a table similar to the one for our spacetime number multiplication.

<table>
<thead>
<tr>
<th>Multiplication</th>
<th>Zero</th>
<th>Positive</th>
<th>Negative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero</td>
<td>Zero</td>
<td>Zero</td>
<td>Zero</td>
</tr>
<tr>
<td>Positive</td>
<td>Zero</td>
<td>Positive</td>
<td>Negative</td>
</tr>
<tr>
<td>Negative</td>
<td>Zero</td>
<td>Negative</td>
<td>Positive</td>
</tr>
</tbody>
</table>

Table 3: Real Number Multiplication

We can, for the sake of simplification, think of a light-like number to be the number zero, a space-like number to be a positive number, and a time-like number to be a negative number. This might be more familiar. Now it becomes easy to see that multiplication of two time-like numbers produces a space-like number, just as two negative numbers produce a positive number when multiplied together. Because light-like numbers behave like zero, it is not possible to divide a space-like number by a light-like number.

Let us look at a more general problem, given two spacetime numbers, such that:

\[ z = A + aj \quad \text{which is a space-like number} \quad A^2 > a^2 \]

\[ Z = b + Bj \quad \text{which is a time-like number} \quad B^2 > b^2 \]

If we multiply the two spacetime numbers together, we obtain:

\[ z \times Z = (A + aj)(b + Bj) \]

\[ = Ab + ABj + abj + aBj^2 \]
\[(Ab+aB) + (AB+ab)j\]

Now let's examine the square of the space part \((Ab+aB)\) of our resulting spacetime number, and the square of the time part \((AB+ab)\).

Space part:

\[(Ab + aB)^2 = (Ab + aB)(Ab + aB)\]

\[= A^2b^2 + AaBb + AaBb + a^2B^2\]

\[= A^2b^2 + 2AaBb + a^2B^2\]

Time part:

\[(AB + ab)^2 = (AB + ab)(AB + ab)\]

\[= A^2B^2 + AaBb + AaBb + a^2b^2\]

\[= A^2B^2 + 2AaBb + a^2b^2\]

Now when we subtract the squares,

\[(Ab + aB)^2 - (AB + ab)^2 = A^2b^2 + 2AaBb + a^2B^2 - (A^2B^2 + 2AaBb + a^2b^2)\]

\[= A^2b^2 + a^2B^2 - A^2B^2 - a^2b^2\]

\[= A^2b^2 - A^2B^2 + a^2B^2 - a^2b^2\]

\[= (A^2b^2 - A^2B^2) - (a^2b^2 - a^2B^2)\]

\[= A^2(b^2 - B^2) - a^2(b^2 - B^2)\]

\[= (A^2 - a^2)(b^2 - B^2)\]

\[A^2 - a^2 > 0 \quad \text{because} \quad A^2 > a^2, \quad \text{and} \quad b^2 - B^2 < 0 \quad \text{because} \quad B^2 > b^2.\]

Therefore, the product is negative, i.e. \(x^2 - t^2 < 0\), which is time-like.

So we have demonstrated that a space-like number multiplied by a time-like
number will yield a time-like number. As an exercise, the reader is asked to show that two time-like numbers produce a space-like number.

Readers having some familiarity with Abstract Algebra might recognize that the spacetime numbers do not form a field like the complex number system. This is because there exists more than one number that do not possess inverses. They are the light-like numbers. It turns out that the spacetime number system is an example of a ring.

5. Zeroes of Polynomials.

Suppose we are given a polynomial such as \( x^2 - 4x + 3 = 0 \).

We can solve this to obtain \( x = 2 \pm 1 \). This corresponds to \( x = 3 \) and \( x = 1 \).

These are considered to be the real roots or zeroes of the polynomial \( x^2 - 4x + 3 = 0 \). Upon closer examination of \( x = 2 \pm 1 \), we can extract \( x = 2 + j \) and \( x = 2 - j \). These two values when substituted back into the polynomial, produce zero, which means that these values of \( x \), given in terms of \( j \), are zeroes or roots of the polynomial as well as \( x = 3 \) and \( x = 1 \). Let's check \( x = 2 + j \).

\[
x^2 - 4x + 3 = 0
\]

\[
(2 + j)^2 - 4(2 + j) + 3 = 0
\]

\[
(2 + j)(2 + j) - 4(2 + j) + 3 = 0
\]

\[
4 + 2j + 2j + j^2 - 8 - 4j + 3 = 0
\]

\[
4 + 4j + 1 - 8 - 4j + 3 = 0
\]

\[
0 = 0
\]

Thus, \( x = 2 + j \) is a zero of the polynomial \( x^2 - 4x + 3 = 0 \). The reader can verify
that \( x = 2 - j \) is also a zero of the above polynomial.

What this means is that in the spacetime plane, there are actually four roots to a quadratic equation. For simplicity, if we take any polynomial which possesses only real roots, no complex roots, it becomes easy to calculate the remaining spacetime roots. All we have to do is form a graph with \( x \) being plotted on the horizontal and \( j \) plotted on the vertical. Next we plot all the real roots on the graph. Then we draw lines with slope equal to +1 and -1 through the plotted root points. The points where these lines intersect corresponds to the remaining spacetime roots of the polynomial. This is represented in Fig.4a. The filled in circles represent the real roots \( x = 1 \) and \( x = 3 \) of the polynomial \( x^2 - 4x + 3 = 0 \). The unfilled circles represent the remaining spacetime roots \( x = 2 + j \) and \( x = 2 - j \).

![Fig 4a. Spacetime Roots of \( x^2 - 4x + 3 = 0 \)](image)

We can extend this idea to a fourth degree polynomial, which has only real roots. We find the rest of the spacetime roots by simply drawing the lines with slope equal to +1 and slope equal to -1 and look to see where these lines intersect. We are left with the situation depicted in Fig 4b.
We have shown that there are four spacetime roots for a quadratic polynomial and sixteen spacetime roots for a quartic polynomial. It can be verified that for any polynomial \(0 = (x-r_1)(x-r_2)(x-r_3)\cdots(x-r_n)\) which possesses \(n\) real roots, there exists \(n^2\) spacetime roots.
Chapter 2  The Spacetime Exponential

2.1 Spacetime Version of Euler’s Formula

Graduate math students should be aware of Euler’s Formula:

\[ e^{i\theta} = \cos \theta + i \sin \theta \]

It takes a complex number and defines it as a vector rotated through an angle \( \theta \).

Since we have seen similarities between the complex number system and the spacetime number system, it makes sense to try to find a spacetime version of Euler’s Formula.

We begin with the power series expansion of \( e^x \):

\[ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \]

If we set \( x = j\alpha \) and substitute into the above power series, we obtain

\[ e^{j\alpha} = 1 + \frac{j\alpha}{1!} + \frac{(j\alpha)^2}{2!} + \frac{(j\alpha)^3}{3!} + \frac{(j\alpha)^4}{4!} + \cdots \]

\[ = 1 + \frac{j\alpha}{1!} + \frac{\alpha^2}{2!} + \frac{j\alpha^3}{3!} + \frac{\alpha^4}{4!} + \cdots \]

\[ = (1 + \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} + \cdots) + j(\frac{\alpha}{1!} + \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} + \cdots) \]

\[ e^{j\alpha} = \cosh \alpha + j \sinh \alpha \]
It is at this point in the study of Complex Analysis that we represent a complex number $x + iy$ in the form of an exponential, $re^{i\theta}$. From this we obtain the following:

1. $r^2 = x^2 + y^2$
2. $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
3. $x = r \cos \theta$
4. $y = r \sin \theta$

Following the same reasoning, if we try to represent our spacetime number $x + jt$ in the form $se^{ja}$, we obtain,

5. $x + jt = scosh \alpha + jsinh \alpha$

Since $\cosh \alpha$ is greater than $\sinh \alpha$ for all $\alpha$, the quantity $scosh \alpha$, which is the space part will be larger than the quantity $jsinh \alpha$, which is the time part. Thus,

6. $x = scosh \alpha$ and $t = ssinh \alpha$

We know that $\cosh^2 \alpha - \sinh^2 \alpha = 1$, so

7. $x^2 - t^2 = s^2$

or,

8. $s = \pm \sqrt{x^2 - t^2}$

We will call $s$ the spacetime interval of the number $z = x + jt$. It makes sense to say that the positive square root corresponds to the positive space-like numbers.

Representing time-like numbers requires a slight modification of the exponential representation of the spacetime number. All we have to do is multiply the quantity on the right hand side of equation (5) by $j$ and it produces,
(9) \( s \sinh \alpha + js \cosh \alpha \)

Now equating this to our space and time parts,

(10) \( x + jt = s \sinh \alpha + js \cosh \alpha \)

Since \( \cosh \alpha \) is greater than \( \sinh \alpha \) for all \( \alpha \), the time part will always be larger than the space part. We can see that \( x = s \sinh \alpha \) and \( y = s \cosh \alpha \) and therefore,

(11) \( s = \pm \sqrt{t^2 - x^2} \)

We now have an equation for space-like numbers

(12) \( z = se^{ia} \)

and an equation representing time-like numbers,

(13) \( z = js e^{ia} \)

We can use the familiar addition formulas for \( \cosh(\alpha + \beta) \) and \( \sinh(\alpha + \beta) \) to see,

(14) \( e^{ia} e^{j\beta} = e^{i(\alpha + \beta)} \)

which is what we should expect. Using equations (12), (13), and (14), it becomes easy to demonstrate that multiplying a space-like number and a time-like number always produces a time-like number.

In complex analysis, we see that the angle \( \theta \) is periodic with a period of \( 2\pi \).

What this meant was that for each complex number \( z = x + iy \), there were infinitely many choices for \( \theta \), since all we had to do was to add \( 2\pi \) in order to obtain another correct representation of the same complex number being considered. This is not the case in the spacetime number system. Recall that if \( z = x + jt \) is a space-like number, it can be represented by \( z = se^{ia} \). It turns out that there is just one \( s \) and one \( \alpha \) that will work. We will now attempt to show this feature.
In complex analysis, the equation \( z = se^{i\theta} \) traces out a unit circle if we take \( s = 1 \) as the angle \( \theta \) varies from 0 to \( 2\pi \). If we use any other values for \( s \), we would produce circles of different radii. If we graph constant values of \( \theta \), we see radial lines emanating from the origin. What happens in the case of the spacetime number system?

2.2 The Unit Space and Unit Time Hyperbolae

In spacetime analysis, we have the equation \( z = e^{i\alpha} \) where \( \alpha \) varies from \( +\infty \) to \( -\infty \). This turns out to be the positive unit hyperbola shown in Fig. 5. Every space-like number with a spacetime interval equal to one is on this unit space unit hyperbola.

Fig. 5: Positive Unit Space Hyperbola

Likewise, every space-like number with a spacetime interval equal to \(-1\) lies on the negative space hyperbola (Fig. 6).

Fig. 6: Negative Unit Space Hyperbola
Consider time-like numbers, $z = \pm je^{ja}$. They have the positive and negative unit time hyperbolae show in Fig. 7.

Spacetime numbers of the form $z = se^{ia}$ are actually hyperbolae of constant interval $s$ whose intercept with the space-axis is the value $s$, see Fig. 8 for an example of a hyperbola with constant space interval equal to 2.
Spacetime numbers of the form $z = jse^{ia}$ are hyperbolae of constant interval $s$ whose intercept with the time-axis is the value $s$. An example of a hyperbola with constant interval $s$ is shown in Fig. 9.

```
Fig. 9: Hyperbola of Constant Time Interval Equal to 2
```

Fig. 10 shows hyperbolae of various constant space and time intervals.

```
Fig. 10: Space and Time Hyperbolae of Constant Intervals
```
We now will attempt to show that values of constant hyperbolic angle $\alpha$ also produce radial lines emerging from the origin, see Fig. 11.

Fig. 11: Hyperbolae of Constant Hyperbolic Angle $\alpha$

Examining Fig. 11, we can see that

\[ P_1 = e^{j\alpha} \]
\[ = \cosh \alpha + j \sinh \alpha \]
\[ = x_1 + t_1 \]
\[ (15) \]

and likewise,

\[ P_2 = 2e^{j\alpha} \]
\[ = 2\cosh \alpha + j2 \sinh \alpha \]
\[ = x_2 + t_2 \]
\[ (16) \]
\[ (17) \]
Using (15), (16) and (17), we can solve to obtain,

\[ x_2 = 2x_1 \]

and

\[ t_2 = 2t_1 \]

Now it becomes obvious that hyperbolae of constant angle \( \alpha \) produce radial lines emanating from the origin.
Chapter 3  Spacetime Functions

3.1 Definitions of Cosh z and Sinh z

In the study of the real calculus, we frequently use the notation $y = f(x)$, with the understanding that $x$ is the independent variable and $y$ is the dependent variable.

In our spacetime calculus, we shall write $w = f(z)$ where $z = x + j t$ is the independent variable and $w = u + j v$ is the dependent variable ($x, t, u$ and $v$ are all real variables).

Let's examine $w = z^2$. We can replace $z$ by $x + j t$ and square it to obtain

$$w = (x + j t)^2 = x^2 + j 2 x t + j^2 t^2$$

$$= (x^2 + t^2) + j (2 x t)$$

Similarly, $w = z^3$ yields $(x^3 + 3 x t^2) + j (3 x^2 t + t^3)$. We can generate $w = z^n$ for any real $n$ in the same fashion.

Using the spacetime version of Euler's Formula,

$$e^{i \alpha} = \cosh \alpha + j \sinh \alpha$$

it seems natural to write,

$$e^z = e^{(x + j t)} = e^x e^{j t}$$

$$= e^x (\cosh t + j \sinh t)$$

This last expression will be our definition of the exponential function.

We now try to find definitions of $\cosh \alpha$ and $\sinh \alpha$.

(1)  $e^{i \alpha} = \cosh \alpha + j \sinh \alpha$
replacing $\alpha$ by $-\alpha$, we have

\[(2) \quad e^{-j\alpha} = \cosh(-\alpha) + j \sinh(-\alpha)\]

and we know $\cosh(-\alpha) = \cosh(\alpha)$ and $\sinh(-\alpha) = -\sinh(\alpha)$, then (2) becomes

\[(3) \quad e^{-j\alpha} = \cosh \alpha - j \sinh \alpha\]

adding (1) and (3), we get

\[e^{j\alpha} + e^{-j\alpha} = \cosh \alpha + j \sinh \alpha + \cosh \alpha - j \sinh \alpha\]

\[= 2 \cosh \alpha\]

which becomes

\[(4) \quad \cosh \alpha = \frac{e^{j\alpha} + e^{-j\alpha}}{2}\]

subtracting (3) from (1) yields

\[e^{j\alpha} - e^{-j\alpha} = \cosh \alpha + j \sinh \alpha - \cosh \alpha + j \sinh \alpha\]

\[= j2 \sinh \alpha\]

\[(5) \quad \sinh \alpha = \frac{e^{j\alpha} - e^{-j\alpha}}{2}\]

Formulas (4) and (5) are natural expressions because they have evolved from standard manipulations of the spacetime version of Euler’s Formula. We can replace the real $\alpha$ in these two expressions by our spacetime $z$.

\[(6) \quad \cosh z = \frac{e^{jz} + e^{-jz}}{2}\]

\[(7) \quad \sinh z = \frac{e^{jz} - e^{-jz}}{2}\]

We could use (6) to find the space part and time part of $\cosh z$ which are spacetime functions $u(x,t)$ and $v(x,t)$ such that $\cosh z = u + jv$, but this is tedious.
There is an alternate way to calculate \( u(x, t) \) and \( v(x, t) \). Earlier we stated that there will be times when we can think of \( j \) as being +1 and -1, now is one of those times.

If we think of \( f(z) = f(x + j t) = u(x, t) + j v(x, t) \) and replacing \( j \) by +1 and -1, we generate the following,

\[
\begin{align*}
(8) & \quad \text{for } j = 1 \quad f(x + t) = u(x, t) + v(x, t) \\
(9) & \quad \text{for } j = -1 \quad f(x - t) = u(x, t) - v(x, t)
\end{align*}
\]

If we add (8) and (9), we have

\[
f(x + t) + f(x - t) = 2u(x, t)
\]

which becomes,

\[
(10) \quad u(x,t) = \frac{f(x + t) + f(x - t)}{2}
\]

If we subtract (9) from (8), we have

\[
f(x + t) - f(x - t) = 2v(x, t)
\]

which becomes,

\[
(11) \quad v(x,t) = \frac{f(x + t) - f(x - t)}{2}
\]

We can use (10) and (11) to calculate the space part and time part of any of the familiar functions from calculus and see what they become in our spacetime calculus.

Let’s examine \( \cosh z \).

\[
\cosh z = u(x, t) + j v(x, t)
\]

Using (10) to solve for the space part of \( \cosh z \),

\[
u(x,t) = \frac{\cosh(x + t) + \cosh(x - t)}{2}
\]
\[
\frac{\cosh x \cosh t + \sinh x \sinh t}{2} + \frac{\cosh x \cosh t - \sinh x \sinh t}{2} = \cosh x \cosh t
\]

Using (11) to solve for the time part of \( \cosh z \),

\[
v(x,t) = \frac{\cosh(x + t) - \cosh(x - t)}{2} = \frac{\cosh x \cosh t + \sinh x \sinh t}{2} - \frac{\cosh x \cosh t - \sinh x \sinh t}{2} = \sinh x \sinh t
\]

Therefore, combining the space and time parts, we get the formula for \( \cosh z \)

(12) \( \cosh z = \cosh x \cosh t + \sinh x \sinh t \)

Let's consider \( \sinh z \) in the same fashion,

\[
\sinh z = u(x,t) + j(x,t)
\]

Solving for \( u(x,t) \) first, then \( v(x,t) \),

\[
u(x,t) = \frac{f(x + t) + f(x - t)}{2} = \frac{\sinh x \cosh t + \cosh x \sinh t}{2} + \frac{\sinh x \cosh t - \cosh x \sinh t}{2} = \sinh x \cosh t
\]

\[
v(x,t) = \frac{\sinh x \cosh t + \cosh x \sinh t}{2} + \frac{\sinh x \cosh t - \cosh x \sinh t}{2} = \cosh x \sinh t
\]

Now combining the space and time parts, we obtain the formula for \( \sinh z \),

(13) \( \sinh z = \sinh x \cosh t + j \cosh x \sinh t \)
3.2 More Trigonometric Formulas in Spacetime

Let's now find the formula for \( \cos z \),

\[
\cos z = u(x,t) + jv(x,t)
\]

\[
u(x,t) = \cos(x + t) + \cos(x - t)
\]

\[
= \frac{\cos x \cos t - \sin x \sin t}{2} + \frac{\cos x \cos t + \sin x \sin t}{2}
\]

\[
= \cos x \cos t
\]

\[
\nu(x,t) = \cos(x + t) - \cos(x - t)
\]

\[
= \frac{\cos x \cos t - \sin x \sin t}{2} + \frac{\cos x \cos t + \sin x \sin t}{2}
\]

\[
= -\sin x \sin t
\]

Therefore,

\[
(14) \quad \cos z = \cos x \cos t - j \sin x \sin t
\]

We now look at \( \sin z \),

\[
\sin z = u(x,t) + jv(x,t)
\]

\[
u(x,t) = \sin(x + t) + \sin(x - t)
\]

\[
= \frac{\sin x \cos t + \cos x \sin t}{2} + \frac{\sin x \cos t - \cos x \sin t}{2}
\]

\[
= \sin x \cos t
\]

\[
\nu(x,t) = \sin(x + t) - \sin(x - t)
\]

\[
= \frac{\sin x \cos t + \cos x \sin t}{2} - \frac{\sin x \cos t - \cos x \sin t}{2}
\]

\[
= \cos x \sin t
\]
Combining the space and time parts, we obtain the formula for \( \sin z \),

\[
(15) \quad \sin z = \sin x \cos t + j \cos x \sin t
\]

From the formulas for \( \cosh z \) and \( \sinh z \), we can generate any of the other hyperbolic trigonometric formulas, such as, \( \tanh z \), \( \text{sech} z \), \( \text{csch} z \), and \( \coth z \). In the same fashion, we can obtain \( \tan z \), \( \sec z \), \( \csc z \), and \( \cot z \).

### 3.3 Wave-Analyticity and The Standing Wave Equation

In complex analysis, we talk about analyticity and having functions satisfy the Cauchy-Riemann Equations. It turns out that there are corresponding concepts, which are wave-analyticity and having functions satisfy the Wave Equation.

Assuming \( f(z) \) is given in terms of its space part and its time part,

\[
f(z) = u(x,t) + jv(x,t)
\]

Then taking a first partial derivative yields,

\[
f'(z) = \frac{\partial u}{\partial x} + j \frac{\partial v}{\partial x}
\]

\[
= u_x + jv_x \quad \text{if} \; z \; \text{is a space-like number}
\]

\[
f'(z) = \frac{\partial u}{\partial t} + j \frac{\partial v}{\partial t}
\]

\[
= v_t + ju_t \quad \text{if} \; z \; \text{is a time-like number}
\]

Equating the space parts, and equating the time parts,

\[
(17) \quad u_x = v_t
\]

\[
(18) \quad u_t = v_x
\]

These last two equations, (17) and (18) are known as the Standing Wave Equations

Taking the partial derivative of (17) with respect to \( x \) and (18) with respect to \( t \), we obtain,
\[ \frac{\partial u_x}{\partial x} = \frac{\partial v_x}{\partial x} \] and
\[ \frac{\partial u_y}{\partial t} = \frac{\partial v_y}{\partial t} \]

which implies,

(19) \quad u_{xx} = v_{xx} \quad \text{and}
(20) \quad u_{tt} = v_{tt}

Since we know that \( v_{tt} = v_{xx} \), equations (19) and (20) imply that

(21) \quad u_{xx} = v_{tt}

We can show the same thing for \( v \),

(22) \quad v_{xx} = v_{tt}

You might notice that these two equations (21) and (22) satisfy the one-dimensional wave equation.
Appendix I

Hyperbolic Rotations

In Calculus, we usually encounter the concept of hyperbolic angles and their associated trigonometric functions, but the idea usually isn’t presented in an easy straightforward manner. It is the purpose of this section to make clear the notion of hyperbolic rotations.

Circular rotations centered at the origin are very simple. All we have to be given is the length of the vector (radius of the associated circle) and the angle that we rotate about the origin.

If we construct the sector from \( -\frac{\pi}{4} < \theta < \frac{\pi}{4} \) represented in Fig. 12a and rotate it through the angle \( \frac{\pi}{4} \), we have the sector now positioned as in Fig. 12b.

This is exactly what we expect from the given circular rotation through the angle \( \frac{\pi}{4} \).
A hyperbolic rotation of the same sector that was given in Fig. 12a through the hyperbolic angle of $\log 2$ is shown in the following Fig. 13.

Figure 13: Hyperbolic Rotation Through Angle of Log 2

From this, we can see that the sector has not rotated at all, but has been stretched in one direction and compressed in the other. We will now attempt to see exactly how much it had been stretched and compressed.

Suppose to each point, $(x, y)$, of the familiar plane, we apply a transformation $x = \frac{x}{k}$ and $y = ky$ so that the point being considered would become $\left(\frac{x}{k}, ky\right)$. It is obvious that each value for $x$ in our familiar plane is compressed by the factor $\frac{1}{k}$ while each corresponding value of $y$ is stretched by the factor $k$. The following Figs. 14a, 14b, 14c show the transformation for $k = 2$ applied to the first quadrant.
We now consider any equilateral hyperbola given by \( y = \frac{a}{x} \). Consider the point \((x_1, y_1)\) on the given hyperbola. Replacing \( y_1 \) by \( y_2 = ky_1 \) and \( x_1 \) by \( x_2 = \frac{x_1}{k} \) yields the equation \( y_2 = \frac{a}{x_2} \), which is still on the same hyperbola. We have just performed a hyperbolic rotation. The actual hyperbolic angle through which we have rotated is \( \alpha = \log k \).
When a circular rotation is performed through an angle $\theta$, it transforms all points on a radial line centered at the origin into points on a new radial line also centered at the origin as shown in Fig. 15a. It is also true for hyperbolic rotations, see Fig. 15b.

It is easy to show that in circular rotations, the angle $\theta$ is equal to two times the area of the sector subtended by the angle $\theta$. Let's consider a unit circle as in Fig. 16.
We know that the area of the circle is given by \( A = \pi r^2 \) and in the case of the unit circle, \( r = 1 \), so \( A = \pi \). The circumference of the unit circle is equal to \( 2\pi \). If we take \( \theta = 2\pi \), we get \( \pi = \frac{\theta}{2} \) and since \( \pi \) is the area of our circle, \( A = \pi = \frac{\theta}{2} \), which implies that \( \theta = 2A \) which in turn shows that \( \theta \) does indeed equal two times the area of the sector subtended by the angle \( \theta \). But, is this true for a hyperbolic rotation through a hyperbolic angle \( \alpha \)? Let's see.

Consider the unit hyperbola given by \( x^2 - y^2 = 1 \). What we are actually trying to see is if \( \alpha \) equals two times the area shown in Fig.17

![Fig.17: Unit Semi-Hyperbola](image)

Recall that the unit circle, \( x^2 + y^2 = 1 \) can be parameterized by the equations \( x = \cos \theta \) and \( y = \sin \theta \). The unit hyperbola, \( x^2 - y^2 = 1 \) can also be parameterized, but by the equations \( x = \cosh \alpha \) and \( y = \sinh \alpha \). Thus, if we have the situation represented by Fig.18.
we know that in terms of $\theta_1$ and $\theta_2$, the area of the sector is given by

$$A = \int_{\theta_1}^{\theta_2} r r d\theta.$$

In order to investigate further, we need to find $d\theta$,

$$\theta = \tan^{-1} \frac{y}{x} \quad \text{where} \quad x = \cosh \alpha \quad \text{and} \quad y = \sinh \alpha$$

Therefore,

$$\theta = \tan^{-1} \left( \frac{\cosh \alpha}{\sinh \alpha} \right)$$

$$\theta = \tan^{-1} (\tanh \alpha)$$

Differentiating this last equation with respect to $\theta$ and substituting $d(tan u) = \left( \frac{1}{1 + u^2} \right) du$

yields,

$$d\theta = \left( \frac{1}{1 + \tanh^2 \alpha} \right) \text{sech}^2 \alpha \, d\alpha$$
\[
\begin{align*}
&= \frac{\text{sech}^2 \alpha}{1 + \tanh \alpha} \, d\alpha \\
&= \frac{d\alpha}{\cosh^2 \alpha + \sinh^2 \alpha}
\end{align*}
\]

Substituting back for \( x = \cosh \alpha \) and \( y = \sinh \alpha \), yields

\[
d\theta = \frac{d\alpha}{x^2 + y^2}
\]

Therefore,

\[
d\theta = \frac{d\alpha}{r^2}.
\]

So,

\[
A = \int_{\alpha_1}^{\alpha_2} \frac{r^2}{2} \left( \frac{d\alpha}{r^2} \right)
\]

in which \( \alpha_1 \) and \( \alpha_2 \) are calculated by

\[
\begin{align*}
\theta &= \tan^{-1}(\tanh \alpha) \\
\tan \theta &= \tanh \alpha \\
\tanh^{-1}(\tan \theta) &= \alpha
\end{align*}
\]

thus,

\[
A = \int_{\alpha_1}^{\alpha_2} d\alpha \\
2A &= \int_{\alpha_1}^{\alpha_2} d\alpha \\
2A &= \alpha_2 - \alpha_1
\]

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Thus, we have shown that twice the area shown in Fig. 18 is indeed equal to the hyperbolic rotation through the hyperbolic angle $\alpha_1 - \alpha_2$. 
Appendix II

Comparing Complex and Spacetime Analysis

The following table is meant to be a quick reference to see the differences as well as the similarities of the Complex Number System and the Spacetime Number System (Reproduced with permission from unpublished seminar notes of T.J. Osler).

<table>
<thead>
<tr>
<th>COMPLEX ANALYSIS</th>
<th>SPACETIME ANALYSIS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Imaginary Unit:</td>
<td>1. Spacetime Unit:</td>
</tr>
<tr>
<td>( i = \sqrt{-1} )</td>
<td>( j = \pm 1 )</td>
</tr>
<tr>
<td>Table of powers:</td>
<td>Table of powers:</td>
</tr>
<tr>
<td>( i^0 = 1 )</td>
<td>( j^0 = 1 )</td>
</tr>
<tr>
<td>( i^1 = \sqrt{-1} )</td>
<td>( j^1 = \pm 1 )</td>
</tr>
<tr>
<td>( i^2 = -1 )</td>
<td>( j^2 = 1 )</td>
</tr>
<tr>
<td>( i^3 = -i )</td>
<td></td>
</tr>
<tr>
<td>( i^4 = 1 )</td>
<td></td>
</tr>
<tr>
<td>( i ) has cycle four</td>
<td>( j ) has cycle two</td>
</tr>
<tr>
<td>2. Complex Number:</td>
<td>2. Spacetime Number:</td>
</tr>
<tr>
<td>( z = x + iy )</td>
<td>( z = x + jt )</td>
</tr>
<tr>
<td>Real Part of ( z ):</td>
<td>Space Part of ( z ):</td>
</tr>
<tr>
<td>( \text{Re} { z } = x )</td>
<td>( \text{Sp} { z } = x )</td>
</tr>
<tr>
<td>Imaginary Part of ( z ):</td>
<td>Time Part of ( z ):</td>
</tr>
<tr>
<td>( \text{Im} { z } = y )</td>
<td>( \text{Ti} { z } = t )</td>
</tr>
<tr>
<td>Light-like Numbers:</td>
<td></td>
</tr>
<tr>
<td>( x^2 = t^2 )</td>
<td></td>
</tr>
<tr>
<td>Space-like Numbers:</td>
<td></td>
</tr>
<tr>
<td>( x^2 &gt; t^2 )</td>
<td></td>
</tr>
<tr>
<td>Time-like Numbers:</td>
<td></td>
</tr>
<tr>
<td>( x^2 &lt; t^2 )</td>
<td></td>
</tr>
</tbody>
</table>
3. Complex Plane

4. Complex Arithmetic:
   Given two arbitrary complex numbers, $Z = X + iY, z = x + iy$.

   Addition:
   $$Z + z = (X + x) + i(Y + y)$$

   Subtraction:
   $$Z - z = (X - x) + i(Y - y)$$

   Multiplication:
   $$Zz = (Xx - Yy) + i(Yx + yX)$$

   Division:
   $$\frac{Z}{z} = \frac{X + iY}{x + iy} \times \frac{x - iy}{x - iy}$$
   $$= \frac{(Xx + Yy) + i(Yx - Xy)}{(x^2 + y^2)}$$

4. Spacetime Plane:

4. Spacetime Arithmetic:
   Given two arbitrary spacetime numbers, $Z = X + jT, z = x + jt$.

   Addition:
   $$Z + z = (X + x) + j(T + t)$$

   Subtraction:
   $$Z - z = (X - x) + j(T - t)$$

   Multiplication:
   $$Zz = (Xx + Tt) + j(Tx + tX)$$

   Division:
   $$\frac{Z}{z} = \frac{X + jT}{x + jt} \times \frac{x - jt}{x - jt}$$
   $$= \frac{(Xx - Tt) + j(Tx - Xx)}{(x^2 - t^2)}$$

   Division is undefined for $x = t = 0$

5. There exists no corresponding topic in the complex number system.

6. The complex numbers form a field

7. Euler's Formula:
   $$e^{i\theta} = \cos \theta + i \sin \theta$$

6. Spacetime numbers form a ring

7. Spacetime Euler’s Formula:
   $$e^{ia} = \cosh \alpha + j \sinh \alpha$$
8. Polar form of $z = x + iy$:

$$z = re^{i\theta}$$

Parametric Equations:

$$x = r \cos \theta$$
$$y = r \sin \theta$$
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

8. Hyperbolic form of $z = x + jt$:

(for space-like $z$):

$$z = se^{ia}$$

$$x = s \cosh \alpha$$
$$y = s \sinh \alpha$$
$$s = \pm \sqrt{x^2 - t^2}$$
$$\alpha = \tanh^{-1}\left(\frac{t}{x}\right)$$

(for time-like $z$):

$$z = jse^{ia}$$

$$x = s \cosh \alpha$$
$$y = s \sinh \alpha$$
$$s = \pm \sqrt{t^2 - x^2}$$
$$\alpha = \coth^{-1}\left(\frac{t}{x}\right)$$

9. Modulus of $z$:

$$|z| = \sqrt{x^2 + y^2}$$

9. Spacetime Interval of $z$:

(for space-like $z$):

Spaceint$\{z\} = \sqrt{x^2 - t^2}$

(for time-like $z$):

Timeint$\{z\} = \sqrt{t^2 - x^2}$

10. Argument of $z$:

$$\text{Arg}\{z\} = \tan^{-1}\left(\frac{y}{x}\right)$$

10. Hyperbolic Argument of $z$:

(for space-like $z$):

Hyparg$\{z\} = \tanh^{-1}\left(\frac{t}{x}\right)$

(for time-like $z$):

Hyparg$\{z\} = \coth^{-1}\left(\frac{t}{x}\right)$
11. Circular Rotations

Multiplication of the vector $z$ by $e^{i\theta}$ rotates the vector through the angle $\theta$ about the origin.

The circular angle $\theta$ is the length of the arc subtended by the unit circle.

The circular angle $\theta$ is also twice the area of the sector subtended by the unit circle.

11. Hyperbolic Rotations

The hyperbolic angle $\alpha$ is twice the area of the sector subtended by the unit semi-hyperbola.

12. Elementary Functions of the form:

$$f(z) = u(x,y) + iv(x,y)$$

$$z^2 = (x^2 - y^2) - t(2xy)$$
$$z^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$$
$$e^z = e^x (\cos y + i \sin y)$$
$$\log z = \log \sqrt{x^2 + y^2} + i \tan^{-1} \left( \frac{y}{x} \right)$$
$$\sin z = \sin x \cosh y + i \cos x \sinh y$$
$$\cos z = \cos x \cosh y + i \sin x \sinh y$$

12. Elementary Functions:

$$f(z) = u(x,t) + jv(x,t)$$

$$z^2 = (x^2 + t^2) + j(2xt)$$
$$z^3 = (x^3 + 3xt^2) + j(3x^2t + t^3)$$
$$e^z = e^x (\cosh t + j \sinh t)$$
$$\log z = \log \sqrt{x^2 - t^2} + j \tanh^{-1} \left( \frac{t}{x} \right)$$

(log $z$ only defined for space-like $z$)

$$\sin z = \sin x \cos t + j \cos x \sin t$$
$$\cos z = \cos x \cos t + j \sin x \sin t$$
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f(z)}{\Delta z} )</td>
<td>A wave-analytic function ( f(z) ) will not require derivatives of all orders. Usually piecewise continuous second derivatives ( (PC_2) ) will suffice.</td>
</tr>
<tr>
<td>where ( \Delta z \to 0 ) in any manner.</td>
<td>( u(x,t) ) and ( v(x,t) ) usually only need piecewise continuous second derivatives.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>14. Differentiation Properties:</th>
<th>15. Cauchy-Riemann Equations:</th>
</tr>
</thead>
<tbody>
<tr>
<td>An analytic function ( f(z) ) has derivatives of all orders.</td>
<td>A function ( f(z) = u(x,y) + iv(x,y) ) is said to be analytic if it satisfies the Cauchy-Riemann Equations:</td>
</tr>
<tr>
<td>( u(x,y) ) and ( v(x,y) ) have partial derivatives of all orders.</td>
<td>( u_x = v_y )</td>
</tr>
<tr>
<td></td>
<td>( u_t = -v_x )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>15. Cauchy-Riemann Equations:</th>
<th>16. Laplace's Equation:</th>
</tr>
</thead>
<tbody>
<tr>
<td>A function ( f(z) = u(x,y) + iv(x,y) ) is said to be analytic if it satisfies the Cauchy-Riemann Equations:</td>
<td>An analytic function satisfies Laplace's Equation if:</td>
</tr>
<tr>
<td></td>
<td>( u_{xx} + u_{yy} = 0 ) and</td>
</tr>
<tr>
<td></td>
<td>( v_{xx} + v_{yy} = 0 )</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>16. Wave Equation:</th>
<th>16. Laplace's Equation:</th>
</tr>
</thead>
<tbody>
<tr>
<td>A wave-analytic function satisfies the Wave Equation if:</td>
<td>An analytic function satisfies Laplace's Equation if:</td>
</tr>
<tr>
<td>( u_{xx} - u_{tt} = 0 ) and</td>
<td>( u_{xx} + u_{yy} = 0 ) and</td>
</tr>
<tr>
<td>( v_{xx} - v_{tt} = 0 )</td>
<td>( v_{xx} + v_{yy} = 0 )</td>
</tr>
</tbody>
</table>
Appendix III

Math-Sci Net Search Results

The following is a list of results using a search of the subject “hyperbolic numbers.” It gives a listing of the author of the original article (AU), the title of the article (YTI), the year published (PY), the journal it was published in (JNO), and a review of the article (YAB).

Record 1 of 7 in MathSci Disc 1998/01-1998/12

AU: Lambert,-Dominique, (B-NDP)

YTI: Les nombres complexes hyperboliques: des complexes qui nous laissent perplexes.

[Hyperbolic complex numbers: complexes that leave us perplexed]

PY: 1995


YAB: The ironical article written by Lambert has two distinguishable kinds of criticisms: (1) General criticisms. These are against a research branch as a whole, "The Hyperbolic Complex Numbers". Then she writes: "The history of hyperbolic complex numbers shows also the difference between what we can call 'deep math', which produces new results because it ramifies on procreative intuitions, and 'shallow math', which is no more than a redundant notation. The theory of functions of one complex variable is a 'deep' theory (its relations with topology, with Leray's sheaf theory... show this in a very eloquent way)...." In the history, this kind of criticism has led even outstanding
scientists to mistakes. We cannot see why it would not lead "not so outstanding scientists" to mistakes as well. Nobody can decree that a research branch has finished or "is not so deep". We note that the complex numbers are related to the Euclidean metric of the plane and therefore to the Euclidean topology employed in it. The hyperbolic numbers would adapt better to a topology related to the 2-dimensional space-time metric. The Euclidean topology in the plane is the topology induced by the Euclidean metric. We cannot speak of a topology induced by the Minkowski metric but we can define a topology closely related to this metric. An example is the so-called "fine topology" introduced by E. C. Zeeman [Topology 6 (1966), 161--170; MR 34\#6699]. In the 2-dimensional space-time the fine topology is the finest topology which induces the 1-dimensional Euclidean topology on every time or space axis. In his article Zeeman writes: "There are four directions in which we suggest the results might be generalized: A theory of 'fine-analytic' functions on Minkowski space. A function is fine-analytic if its restriction to each space and time axis is analytic. Thus a function which is analytic except for singularities in the light cone, may be fine-analytic everywhere." It is possible that the hyperbolic complex numbers are associated to a theory of functions on the "Minkowski plane", with a topology specially adapted to the Minkowski metric. The apparent lack of deep results on the hyperbolic complex numbers may be a consequence of the lack of more research on them, and Lambert's article tends to inhibit this research.

(2) Personal criticisms. The second kind of criticism is personal, and Lambert is very rigorous. Concerning the reason why people (mainly physicists) still publish in this area Lambert says: "They must increase at any price their number of publications for more alimentary than scientific reasons...." Unfortunately Lambert is right. Therefore
alimentation problems are common to all humankind and Lambert's article is not
different from those she criticizes. This makes us remember an old fable of La Fontaine,
"Le Renard et les Raisins", which begins: "Certain Renard Gascon, d'autres disent
Normand, Mourant presque de fain, vit au haut d'une treille Des Raisins murs
apparemment ..."

Record 2 of 7 in MathSci Disc 1998/01-1998/12

AU: Magill,-K.-D., Jr., (1-SUNYB)

YTI: Two-dimensional nonassociative Euclidean nearrings and the ring of hyperbolic
numbers.

PY: 1997

JNO: Publ.-Math.-Debrecen [Universitatis-Debreceniensis.-Institutum-Mathematicum.-

YAB: Summary: "The additive group of the ring H of hyperbolic numbers is isomorphic
to the two-dimensional Euclidean group and the product $v \otimes w$ of two elements $v$ and $w$
is defined by $v \otimes w = (v_1w_1 + v_2w_2, v_1w_2 + v_2w_1)$. The ring H has a nontrivial central
idempotent (in fact, it has exactly two) and it has exactly four central involutions. We
show that each of these properties characterizes H, up to isomorphism, within the class of
all those nonassociative topological nearrings with a left identity whose additive group is
the two-dimensional Euclidean group." In addition the author gives a reference to a very
nice article on hyperbolic numbers by G. Sobczyk [College Math. J. 26 (1995), no. 4,
268--280].
In this paper, the authors make an interesting attempt to develop an analogue of the fractional calculus in the framework of the matrix calculus. The matrix $A^B$, where $A$ and $B$ are square matrices, is defined by making use of eigenvalues and the associated projection matrices. Some related algebraic concepts such as hyperbolic numbers, quaternions and the Klein group are also outlined. Next, the symbol $D^n = \frac{d^n}{dx^n}$ is given an interpretation in the case when the order of the derivative is a square matrix, i.e. the authors define $D^A$, where $A$ is a square matrix. As an application of the derivative of matrix order, a generalization of the Rodrigues formula for Legendre polynomials is also obtained.
YAB: Double complex functions involve the use of a double imaginary unit, J, which may represent either the ordinary imaginary unit, i, or the unit of the dual (or hyperbolic) numbers, where the square is +1. The author's use of functions of J, referred to as double complex functions, allows the creation of a mapping between complex-valued functions and dual-valued functions. The author then defines differential equations that permit related complex-valued solutions and dual-valued ones, referred to as double complex normal equations. Having a double complex function as a solution of a double complex normal equation allows one to have several real-valued solutions, related by the method already referred to. It is shown that the Ernst equation, which can be used to determine solutions of Einstein's vacuum gravitational field equations for stationary, axisymmetric metrics, can be written in the double complex normal form. As a result of this, one can use known real-valued or complex-valued solutions to infer double complex functions as solutions. Having such a solution, the other real parts of it are also solutions to the original real equation. Therefore, this is a technique to generate new solutions from old, popular with the Ernst equation for many years. The author compares his technique to those of Tanabe, of Kramer and Neugebauer, and of Belinskii and Zakharov. In each case, he shows that his technique acquires new solutions previously undetermined. As well, by this method, one finds some inner relationships between solutions found by previous authors.

Record 5 of 7 in MathSci Disc 1988 - 1992

AU: Ren,-Hungshan [Ren,-Hong-Shan]

YTI: The hyperbolic quasifield of numbers on the plane and analytic functions on it.
This paper is motivated by a certain problem of ordinary differential equations. We introduce a quasifield of numbers, hyperbolic quasifield of numbers and other concepts derived from them, mainly to discuss algebraic and analytic properties of a hyperbolic quasifield of numbers.

Two-dimensional nonassociative Euclidean nearrings and the ring of hyperbolic numbers.

Hyperbolic complex Yang-Baxter equation and hyperbolic complex multiparametric quantum groups.
YAB: Hyperbolic numbers are analogous to the complex numbers and contain the additional hyperbolic unit \( e^2 = +1 \). The hyperbolic Yang-Baxter equation is equivalent to a system of two real or complex Yang-Baxter equations. Correspondingly, a hyperbolic quantum group is isomorphic to a direct product of ordinary quantum groups. In particular, a hyperbolic linear quantum group is considered for the multiparametric solution of the hyperbolic Yang-Baxter equation.
Appendix IV

References


